

The mixed Yamabe problem for harmonic foliations

Vladimir Rovenski* and Leonid Zelenko†
Mathematical Department, University of Haifa

Abstract

The mixed scalar curvature of a foliated Riemannian manifold, i.e., an averaged mixed sectional curvature, has been considered by several geometers. We explore the Yamabe type problem: to prescribe the constant mixed scalar curvature for a foliation by a conformal change of the metric in normal directions only. For a harmonic foliation, we derive the leafwise elliptic equation and explore the corresponding nonlinear heat type equation. We assume that the leaves are compact submanifolds and the manifold is fibered instead of being foliated, and use spectral parameters of certain Schrödinger operator to find solution, which is attractor of the equation.

Keywords: foliation, Riemannian metric, harmonic, mixed scalar curvature, biconformal, Schrödinger operator, parabolic PDE, attractor

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Introduction

Geometrical problems of prescribing curvature-like invariants (e.g. the scalar curvature and the mean curvature) of manifolds and foliations are popular for a long time, see [6, 18, 21]. There are many proofs of positive answer to the Yamabe problem: given a closed Riemannian manifold (M, g) of $\dim M \geq 3$, find a metric conformal to g with constant scalar curvature. The study of this geometrical problem was began by Yamabe (1960) and completed by Trudinger, Aubin and Schoen (1986), its solution is expressed in terms of the existence and multiplicity of solutions of a given elliptic PDE in the Riemannian manifold, see [2, 14]. Several authors developed an analog of the problem for CR-manifolds, see [9], and its generalization to contact (real or quaternionic) manifolds. The problem when metrics of constant scalar curvature can be produced on warped product manifolds has been studied in several articles, see [8].

Let (M, g) be endowed with a p -dimensional foliation \mathcal{F} . Denote by $\mathcal{D} = T\mathcal{F}$ the tangent distribution and \mathcal{D}^\perp ($\dim \mathcal{D}^\perp = n$) the orthogonal distribution (or the normal subbundle) of the tangent bundle TM . In [4], a tensor calculus, adapted to the orthogonal splitting

$$TM = \mathcal{D} + \mathcal{D}^\perp \tag{1}$$

is developed to study the geometry of both the distributions and the ambient manifolds. We have $g = g_{\mathcal{F}} + g^\perp$, where $g^\perp(X, Y) := g(X^\perp, Y^\perp)$ and $(\cdot)^\perp$ is the projection of TM onto \mathcal{D}^\perp . Obviously, biconformal metrics $\tilde{g} = v^2 g_{\mathcal{F}} + u^2 g^\perp$ ($u, v > 0$) preserve (1) and extend the class of conformal metrics (i.e., $u = v$). Biconformal metrics (e.g. doubly-twisted products, introduced by Ponge and Reckziegel in [12]) have many applications in differential geometry,

*E-mail: rovenski@math.haifa.ac.il

†E-mail: zelenko@math.haifa.ac.il

relativity, quantum-gravity, etc, see [8]. The \mathcal{D}^\perp - or \mathcal{D} -conformal metrics correspond to $v \equiv 1$ or $u \equiv 1$, see [17]–[20].

Using the natural representation of $O(p) \times O(n)$ on TM , A.M. Naveira [11] distinguished thirty-six classes of Riemannian almost-product manifolds $(M, g, \mathcal{D}, \mathcal{D}^\perp)$; some of them are foliated, e.g., harmonic, totally geodesic, conformal, and Riemannian foliations. Following this line of research, several geometers completed the geometric interpretation and gave examples for each class of almost-product structures. The simple examples of harmonic foliations are geodesic ones (e.g., parallel circles or winding lines on a flat torus, and a Hopf field of great circles on the 3-sphere). H. Rummeler characterized harmonic foliations by existence of an \mathcal{F} -closed differential p -form that is transverse to \mathcal{F} . D. Sullivan’s topological tautness condition is equivalent to the existence of a metric on M making a foliation harmonic, see [6].

The components of the curvature of a foliation can be tangential, transversal, and mixed. The tangential geometry of a foliation is infinitesimally modeled by the tangent distribution to the leaves, while the transversal geometry – by the orthogonal distribution \mathcal{D}^\perp . Prescribing the sign of tangential scalar curvature has been studied for foliated spaces, for example, there is no foliation of positive leafwise scalar curvature on any torus, see [25]. The transversal scalar curvature is well studied for Riemannian foliations, e.g. the “transversal Yamabe problem”, see [23].

The mixed scalar curvature, S_{mix} , for foliated (sub)manifolds has been considered by several geometers, see [3, 13, 22], but its constancy (so called “mixed Yamabe problem”) is less studied. In [19, 20], we prescribed the sign of S_{mix} using flows of \mathcal{D}^\perp -conformal metrics. In this paper we explore the following Yamabe type problem: *Given a harmonic foliation \mathcal{F} of a Riemannian manifold (M, g) , find a \mathcal{D}^\perp -conformal metric \tilde{g} with leafwise constant mixed scalar curvature.* For a general foliation, the topology of the leaf through a point can change dramatically with the point; this gives many difficulties in studying leafwise parabolic and elliptic equations. Therefore, in the paper (at least in the main results) we assume that

$$\mathcal{F} \text{ is defined by an orientable fiber bundle.} \quad (2)$$

The proof of main results is based on Sect. 1.2 (with variation formulae for geometrical quantities under \mathcal{D}^\perp -conformal change of a metric), Sect. 1.3 (with Proposition 3 and Corollary 2), Sect. 2 (about attractor of the nonlinear heat equation on a closed manifold and about solution of its stationary equation with parameter) and Sect. 3 (about smooth dependence of a solution on a transversal parameter).

A slight change in the proof allows us to extend the main results for the case when the prescribed mixed scalar curvature is not leafwise constant.

1 Main results

The main results of the paper are the following.

Theorem 1. *Let \mathcal{F} be a harmonic and nowhere totally geodesic foliation of a closed Riemannian manifold (M, g) with condition (2). Then there exists a \mathcal{D}^\perp -conformal metric \tilde{g} with leafwise constant mixed scalar curvature.*

If \mathcal{D}^\perp is integrable than Corollary 2 (see Sect. 1) is applicable. In particular case of codimension-one foliations, we have the following.

Corollary 1. *Let \mathcal{F} be a codimension-one harmonic and nowhere totally geodesic foliation of a Riemannian manifold (M, g) with condition (2). Then there exists a \mathcal{D}^\perp -conformal metric \tilde{g} with leafwise constant Ricci curvature in the normal direction.*

There are examples of foliations of codimension > 1 with minimal, not totally geodesic leaves on (compact) Lie groups with left-invariant metrics, see [24]; further, the metric can be chosen to be bundle-like with respect to \mathcal{F} . Such foliations have leafwise constant mixed scalar curvature.

Theorem 2. *Let \mathcal{F} ($\dim \mathcal{F} > 1$) be a totally geodesic foliation of a closed Riemannian manifold (M, g) with condition (2) and integrable normal distribution. Then there exists a \mathcal{D}^\perp -conformal metric \tilde{g} with leafwise constant mixed scalar curvature.*

1.1. Preliminaries. Denote by $R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X, Y]}$ the curvature tensor of Levi-Civita connection. The sectional curvature $K(X, Y) = g(R(X, Y)X, Y)$, where $X \in T\mathcal{F}$, $Y \in \mathcal{D}^\perp$ are unit vectors, is called *mixed*. It regulates (through the Jacobi equation) the deviation of leaves along the leaf geodesics. Foliation with constant mixed sectional curvature play an important role in differential geometry, but are far from being classified. Examples are k -nullity foliations on Riemannian manifolds which are totally geodesic, relative nullity foliations, which determine a ruled structure of submanifolds in space forms, foliations produced by Reeb vector field on Sasakian manifolds, etc. Totally geodesic foliations on complete manifolds with $K_{\text{mix}} \equiv 0$ split. For a k -dimensional totally geodesic foliation with $K_{\text{mix}} \equiv 1$ on a closed manifold M^{n+k} , we have the Ferus's inequality $k < \rho(n)$, where $\rho(n) - 1$ is the number of linear independent vector fields on a sphere S^{n-1} , see [16].

The *mixed scalar curvature* is an averaged mixed sectional curvature,

$$S_{\text{mix}} = \sum_{j=1}^n \sum_{a=1}^p K(\mathcal{E}_j, E_a),$$

and is independent of the choice of a local orthonormal frame $\{\mathcal{E}_j, E_a\}_{j \leq n, a \leq p}$ of TM adapted to \mathcal{D}^\perp and $T\mathcal{F}$, see [15, 16, 22]. If either \mathcal{D}^\perp or $T\mathcal{F}$ is one-dimensional and tangent to a unit vector field N , then S_{mix} is the Ricci curvature in the N -direction.

Let \mathfrak{X}_M be the module over $C^\infty(M)$ of all vector fields on M , and \mathfrak{X}^\perp and \mathfrak{X}^\top the modules of all vector fields on \mathcal{D}^\perp and $T\mathcal{F}$, respectively. The extrinsic geometry of a foliation is related to the second fundamental form of the leaves, $h(X, Y) = (\nabla_X Y)^\perp$, where $X, Y \in \mathfrak{X}^\top$, and its invariants (e.g., the mean curvature $H = \text{Tr}_g h$). Special classes of foliations such as totally geodesic, $h = 0$ (with the simplest extrinsic geometry); totally umbilical, $h = (H/p)g_{\mathcal{F}}$; and harmonic, $H = 0$, have been studied by many geometers, see survey in [16]. Let h^\perp be the second fundamental form of \mathcal{D}^\perp , $H^\perp = \text{Tr}_g h^\perp$ the mean curvature, and T the integrability tensor of \mathcal{D}^\perp . We have

$$2h^\perp(X, Y) = (\nabla_X Y + \nabla_Y X)^\top, \quad 2T(X, Y) = [X, Y]^\top, \quad X, Y \in \mathfrak{X}^\perp. \quad (3)$$

The formula in [22], for foliations reads as

$$S_{\text{mix}} = \|H^\perp\|^2 - \|h^\perp\|^2 + \|T\|^2 + \|H\|^2 - \|h\|^2 + \text{div}(H^\perp + H). \quad (4)$$

We calculate norms of tensors using local adapted basis as

$$\|h^\perp\|^2 = \sum_{i,j} \|h^\perp(\mathcal{E}_i, \mathcal{E}_j)\|^2, \quad \|h\|^2 = \sum_{a,b} \|h(E_a, E_b)\|^2, \quad \|T\|^2 = \sum_{i,j} \|T(\mathcal{E}_i, \mathcal{E}_j)\|^2.$$

Example 1. (Constant mixed scalar curvature on doubly-twisted products). The *doubly twisted product* of Riemannian manifolds $(B, g_{\mathcal{F}})$ and (F, g^\perp) , is a manifold $M = B \times F$ with the metric $g = v^2 g_{\mathcal{F}} + u^2 g^\perp$, where $v, u \in C^\infty(B \times F)$ are positive functions. It is called the *doubly warped product* of $(B, g_{\mathcal{F}})$ and (F, g^\perp) if the warping functions v and u only depend on the points of B and F , respectively.

The *leaves* $B \times \{y\}$ of a doubly-twisted product $B \times_{(v,u)} F$ and the *fibers* $\{x\} \times F$ are totally umbilical. We have

$$h = -(\nabla^\perp \log v) g_{\mathcal{F}}, \quad h^\perp = -(\nabla^\top \log u) g^\perp.$$

By the above, $H = -n \nabla^\perp \log v$, $H^\perp = -p \nabla^\top \log u$, and

$$\|H\|^2 - \|h\|^2 = (n^2 - n) \|\nabla^\perp v\|^2 / v^2, \quad \|H^\perp\|^2 - \|h^\perp\|^2 = (p^2 - p) \|\nabla^\top u\|^2 / u^2.$$

Next we derive

$$\begin{aligned} \operatorname{div} H &= -p (\Delta^\top u) / u - (p^2 - p) \|\nabla^\top u\|^2 / u^2, \\ \operatorname{div} H^\perp &= -n (\Delta^\perp v) / v - (n^2 - n) \|\nabla^\perp v\|^2 / v^2, \end{aligned}$$

where Δ^\top is the leafwise Laplacian and Δ^\perp is the fiberwise Laplacian. Substituting in (4) with $T = 0$, we obtain the formula

$$S_{\text{mix}} = -n (\Delta^\top u) / u - p (\Delta^\perp v) / v.$$

Let B be a closed manifold. Given a positive function $v \in C^\infty(B \times F)$, define a leafwise Schrödinger operator $\mathcal{H} = -\Delta^\top - \beta$, where $\beta = \frac{p}{n} (\Delta^\perp v) / v$. For any compact leaf, the spectrum of \mathcal{H} is discrete, the least eigenvalue λ_0 is isolated from other eigenvalues, and the eigenfunction e_0 (called the ground state) can be chosen positive, see Sect. 2. Since $\mathcal{H}(e_0) = \lambda_0 e_0$, a doubly-twisted product $B \times_{(v,e_0)} F$ has leafwise constant mixed scalar curvature equal to $n\lambda_0$.

By Lemma 2, \mathcal{D}^\perp -conformal changes of the metric preserve harmonic foliations.

We focus on the mixed Yamabe problem for harmonic foliations, which amounts to finding a positive solution of the leafwise elliptic equation, see Proposition 2,

$$-n (\Delta^\top u + \beta^\top u) = -2 H^\perp(u) + \tilde{S}_{\text{mix}} u + \|h\|_g^2 u^{-1} - \|T\|_g^2 u^{-3}, \quad (5)$$

where $\beta^\top = \frac{1}{n} (\|T\|_g^2 - \|h\|_g^2 - S_{\text{mix}})$, and a leafwise constant \tilde{S}_{mix} corresponds to a \mathcal{D}^\perp -conformal metric \tilde{g} . Proposition 1 allows us to reduce (5) to the case of $H^\perp = 0$.

Example 2. The global structure of totally geodesic foliations with integrable normal bundle (i.e., \mathcal{D}^\perp is tangent to a foliation \mathcal{F}^\perp) has been studied in [5]: the universal cover \tilde{M} is topologically a product $\tilde{F} \times \tilde{F}^\perp$ of universal covers of the leaves of both foliations, \mathcal{F} and \mathcal{F}^\perp . Let \mathcal{F} be a totally geodesic foliation with integrable normal bundle of a closed Riemannian manifold (M, g) with conditions (2) and $H^\perp = 0$. Then $\Psi_1 = \Psi_2 = 0$, and (5) becomes the linear elliptic equation on F ,

$$-\Delta^\top u - (\beta^\top + \Phi) u = 0, \quad (6)$$

where $\beta^\top = -\frac{1}{n} S_{\text{mix}}$. Suppose that $S_{\text{mix}} \neq \text{const}$ and $\Phi = \text{const}$. Then $\mathcal{H}(u_*) = \Phi u_*$, where $u_* = e_0$ and $\Phi = \lambda_0$ for the Schrödinger operator $\mathcal{H} = -\Delta^\top - \beta^\top$. Assuming $\nabla^\perp u|_F = 0$, continue u_* smoothly on M . Thus, the mixed scalar curvature of the Riemannian manifold $(M, \tilde{g} = g^\top + u_*^2 g^\perp)$ is $n\Phi$.

Proposition 1. *Let \mathcal{F} be a foliation of a closed Riemannian manifold (M, g) with condition (2). Then there exists a smooth function $u > 0$ on M such that $H^\perp = 0$ for the metric $\tilde{g} = g_{\mathcal{F}} + u^2 g^\perp$.*

Proof. Recall the equality for any $X, Y \in \mathfrak{X}^\perp$ and $U, V \in \mathfrak{X}^\top$, see [16],

$$\begin{aligned} g(R(U, X)V, Y) &= g(((\nabla_U C)_V - C_V C_U)(X), Y) \\ &+ g(((\nabla_X A^\top)_Y - A_X^\top A_Y^\top)(U), V), \end{aligned} \quad (7)$$

where the co-nullity operator $C : T\mathcal{F} \times TM \rightarrow \mathcal{D}^\perp$ is defined by $C_U(X) = -(\nabla_X U)^\perp$ ($U \in \mathfrak{X}^\top$, $X \in \mathfrak{X}_M$). Note that

$$\begin{aligned} \sum_j g((\nabla_U C)_V(\mathcal{E}_j), \mathcal{E}_j) &= \sum_j \nabla_U(g(C_V(\mathcal{E}_j), \mathcal{E}_j)) \\ &= \nabla_U(g(\sum_j h(\mathcal{E}_j, \mathcal{E}_j), V)) = g(\nabla_U H^\perp, V). \end{aligned}$$

Thus, tracing (7) over \mathcal{D} and taking the antisymmetric part, we obtain $d^\top H^\perp = 0$, where the 2-form $d^\top H^\perp$ is defined by

$$2d^\top H^\perp(U, V) = g(\nabla_U H^\perp, V) - g(\nabla_V H^\perp, U) \quad (U, V \in \mathfrak{X}^\top).$$

Then we apply Lemma 1 given below. □

Lemma 1 (see Theorem 1.1 in [18]). *Let \mathcal{F} be a foliation of a closed Riemannian manifold (M, g) with condition (2), and $d^\top H^\perp = 0$. Then the Cauchy's problem*

$$\partial_t g = -(2/p)(\operatorname{div}^\top H^\perp)g^\perp, \quad g_0 = g,$$

has a unique solution g_t ($t \geq 0$) that converges as $t \rightarrow \infty$ to a metric with $H^\perp = 0$.

1.2. \mathcal{D}^\perp -conformal change of a metric. We shall find how various geometrical quantities are transformed under \mathcal{D}^\perp -conformal change of a metric. The Weingarten operator A_U^\perp of \mathcal{D}^\perp and the skew-symmetric operator T_U^\sharp , where $U \in \mathfrak{X}^\top$, are given by

$$g(A_U^\perp(X), Y) = g(h^\perp(X, Y), U), \quad g(T_U^\sharp(X), Y) = g(T(X, Y), U).$$

Lemma 2. *Given a foliation \mathcal{F} on $(M, g = g_\mathcal{F} + g^\perp)$, and $\phi \in C^1(M)$, define a new metric $\tilde{g} = g_\mathcal{F} + e^{2\phi}g^\perp$. Then*

$$\tilde{h}^\top = e^{-2\phi}h, \quad \tilde{H}^\top = e^{-2\phi}H, \quad (8)$$

$$\tilde{h}^\perp = e^{2\phi}(h^\perp - (\nabla^\top \phi)g^\perp), \quad \tilde{H}^\perp = H^\perp - n\nabla^\top \phi, \quad (9)$$

$$\tilde{A}_U^\perp = A_U^\perp - U(\phi)\operatorname{id}^\perp, \quad \tilde{T}_U^\sharp = e^{-2\phi}T_U^\sharp \quad (U \in \mathfrak{X}^\top). \quad (10)$$

Hence, \mathcal{D}^\perp -conformal variations preserve total umbilicity, harmonicity, and total geodesy of \mathcal{F} , and preserve total umbilicity of the normal distribution \mathcal{D}^\perp .

Proof. The *Levi-Civita connection* ∇ of a metric g is given by the known formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &+ g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \quad (X, Y, Z \in \mathfrak{X}_M). \end{aligned} \quad (11)$$

Formula (8)₁ follows from (11):

$$\begin{aligned} 2e^{2\phi}g(\tilde{\nabla}_U V, X) &= 2\tilde{g}(\tilde{\nabla}_U V, X) \\ &= -Xg(U, V) - g([U, X], V) - g([V, X], U) = 2g(\nabla_U V, X). \end{aligned}$$

We deduce (8)₂ using $\tilde{H}^\top = e^{-2\phi} \sum_a h(E_a, E_a) = e^{-2\phi} H$. From $\tilde{T} = T$ and

$$\begin{aligned} g(\tilde{T}_U^\sharp(X), Y) &= e^{-2\phi} \tilde{g}(\tilde{T}_U^\sharp(X), Y) = e^{-2\phi} \tilde{g}(T(X, Y), U) \\ &= e^{-2\phi} g(T(X, Y), U) = e^{-2\phi} g(T_U^\sharp(X), Y) \end{aligned}$$

formula (10)₂ follows. By (11), for any $X, Y \in \mathfrak{X}^\perp$ and $U \in \mathfrak{X}^\top$ we have

$$g(\tilde{\nabla}_X Y, U) = e^{2\phi} g(\nabla_X Y, U) - e^{2\phi} U(\phi) g(X, Y) - (e^{2\phi} - 1) g(T(X, Y), U). \quad (12)$$

From this, skew-symmetry of T and (3), we deduce (9)₁. Then we get (10)₁ using

$$\begin{aligned} e^{2\phi} g(\tilde{A}_U^\perp(X), Y) &= \tilde{g}(\tilde{A}_U^\perp(X), Y) = \tilde{g}(\tilde{h}^\perp(X, Y), U) \\ &= e^{2\phi} (g(A_U^\perp(X), Y) - U(\phi) g(X, Y)). \end{aligned}$$

Similarly, we prove (10)₂:

$$e^{2\phi} g(\tilde{T}_U^\sharp(X), Y) = \tilde{g}(\tilde{T}_U^\sharp(X), Y) = \tilde{g}(\tilde{T}(X, Y), U) = g(T_U^\sharp(X), Y).$$

The orthonormal bases of \mathcal{D}^\perp in both metrics are related by $\tilde{\mathcal{E}}_j = e^{-\phi} \mathcal{E}_j$. To show this we calculate for any $j \leq n$,

$$1 = \tilde{g}(\tilde{\mathcal{E}}_j, \tilde{\mathcal{E}}_j) = e^{2\phi} g(e^{-\phi} \mathcal{E}_j, e^{-\phi} \mathcal{E}_j) = g(\mathcal{E}_j, \mathcal{E}_j). \quad (13)$$

By (9)₁, we have

$$\tilde{h}^\perp(\tilde{\mathcal{E}}_j, \tilde{\mathcal{E}}_j) = e^{-2\phi} \tilde{h}^\perp(\mathcal{E}_j, \mathcal{E}_j) = h^\perp(\mathcal{E}_j, \mathcal{E}_j) - (\nabla^\top \phi) g(\mathcal{E}_j, \mathcal{E}_j).$$

From this and the definition $H^\perp = \text{Tr}_g h^\perp$, the equality (9)₂ follows. \square

Remark 1. By Lemma 2, for a leafwise constant ϕ we obtain $\tilde{h}^\perp = e^{2\phi} h^\perp$ and $\tilde{H}^\perp = H^\perp$. Hence, \mathcal{D}^\perp -scalings of g preserve harmonicity and total geodesy of \mathcal{D}^\perp .

Proposition 2. *The mixed scalar curvature of a harmonic foliation \mathcal{F} under \mathcal{D}^\perp -conformal change of a metric $\tilde{g} = g_{\mathcal{F}} + u^2 g^\perp$, where $u > 0$ is a smooth function, is transformed according to the formula*

$$(\text{S}_{\text{mix}} - \tilde{\text{S}}_{\text{mix}}) u = n \Delta^\top u - 2 H^\perp(u) + \|h\|_g^2 (u^{-1} - u) - \|T\|_g^2 (u^{-3} - u). \quad (14)$$

If, in particular, u is leafwise constant (i.e., \tilde{g} is a \mathcal{D}^\perp -scaling of g), then we have

$$\tilde{\text{S}}_{\text{mix}} = \text{S}_{\text{mix}} - (u^{-2} - 1) \|h\|_g^2 + (u^{-4} - 1) \|T\|_g^2.$$

Proof. By Lemma 2, we have

$$\begin{aligned} \|\tilde{h}^\top\|_g^2 &= e^{-2\phi} \|h\|_g^2, \quad \|\tilde{T}\|_g^2 = e^{-4\phi} \|T\|_g^2, \quad \|\tilde{h}^\perp\|_g^2 = \|h^\perp\|_g^2 + n \|\nabla^\top \phi\|_g^2 - 2 H^\perp(\phi), \\ \|\tilde{H}^\perp\|_g^2 &= \|H^\perp\|_g^2 + n^2 \|\nabla^\top \phi\|_g^2 - 2 n H^\perp(\phi), \quad \widetilde{\text{div}}^\top \tilde{H}^\perp = \text{div}^\top H^\perp - n \Delta^\top \phi. \end{aligned} \quad (15)$$

Indeed, the formulae for $\|\tilde{h}^\top\|_g^2$ and $\|\tilde{T}\|_g^2$ follow from

$$\begin{aligned} \|\tilde{h}^\top\|_g^2 &= \sum_{a,b,i} \tilde{g}(\tilde{h}^\top(E_a, E_b), \tilde{\mathcal{E}}_i)^2 = e^{4\phi} \sum_{a,b,i} g(e^{-2\phi} h(E_a, E_b), e^{-\phi} \mathcal{E}_i)^2 \\ &= e^{-2\phi} \sum_{a,b,i} g(h(E_a, E_b), \mathcal{E}_i)^2 = e^{-2\phi} \|h\|_g^2, \\ \|\tilde{T}\|_g^2 &= \sum_{a,b,i} \tilde{g}(\tilde{T}(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_j), E_a)^2 = \sum_{a,b,i} g(T(e^{-\phi} \tilde{\mathcal{E}}_i, e^{-\phi} \tilde{\mathcal{E}}_j), E_a)^2 \\ &= e^{-4\phi} \sum_{a,b,i} g(T(\mathcal{E}_i, \mathcal{E}_j), E_a)^2 = e^{-4\phi} \|T\|_g^2. \end{aligned}$$

Formula for $\widetilde{\operatorname{div}}^\top \tilde{H}^\perp$ follows from $g(\tilde{\nabla}_a U, E_a) = g(\nabla_a U, E_a)$ for $U \in \mathfrak{X}^\top$ and

$$\widetilde{\operatorname{div}}^\top \tilde{H}^\perp = \sum_a \tilde{g}(\tilde{\nabla}_a \tilde{H}^\perp, E_a) \operatorname{div}^\top H^\perp - n \operatorname{div}^\top (\nabla^\top \phi).$$

From

$$\begin{aligned} \|\tilde{h}^\perp\|_g^2 &= \sum_{a,i,j} \tilde{g}(\tilde{h}^\perp(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_i), E_a)^2 = \sum_{a,i,j} (g(h^\perp(\mathcal{E}_i, \mathcal{E}_i) - (\nabla^\top \phi) g(\mathcal{E}_i, \mathcal{E}_i), E_a))^2 \\ &= \|h^\perp\|_g^2 - 2g(H^\perp, \nabla^\top \phi) + n \|\nabla^\top \phi\|_g^2, \\ \|\tilde{H}^\perp\|_g^2 &= g(\tilde{H}^\perp, \tilde{H}^\perp) = g(H^\perp - n\nabla^\top \phi, H^\perp - n\nabla^\top \phi) \\ &= \|H^\perp\|_g^2 - 2ng(H^\perp, \nabla^\top \phi) + n^2 \|\nabla^\top \phi\|_g^2 \end{aligned}$$

the formulae for $\|\tilde{h}^\perp\|_g^2$ and $\|\tilde{H}^\perp\|_g^2$ follow. Then, using (15), $\tilde{\mathcal{E}}_i = e^{-\phi} \mathcal{E}_i$, and

$$\tilde{S}_{\text{mix}} = \sum_i \tilde{r}(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_i) = e^{-2\phi} \sum_i \tilde{r}(\mathcal{E}_i, \mathcal{E}_i),$$

we obtain the formula

$$\tilde{S}_{\text{mix}} = S_{\text{mix}} - n(\Delta^\top \phi + \|\nabla^\top \phi\|_g^2) + 2H^\perp(\phi) + (e^{-4\phi} - 1)\|T\|_g^2 - (e^{-2\phi} - 1)\|h\|_g^2. \quad (16)$$

Substituting $\phi = \log u$ and $\nabla^\top \phi = u^{-1} \nabla^\top u$, $\Delta^\top \phi = u^{-1} \Delta^\top u - u^{-2} \|\nabla^\top u\|_g^2$ into (16) yields the required formula (14), which is equivalent to (5). \square

1.3. Proof of the main results. Proposition 1 allows us to assume $H^\perp = 0$. Then we associate with (5) the leafwise parabolic equation with a leafwise constant \tilde{S}_{mix}

$$\partial_t u - \Delta^\top u - (\beta^\top + \tilde{S}_{\text{mix}}/n)u = (\|h\|^2/n)u^{-1} - (\|T\|^2/n)u^{-3}. \quad (17)$$

We shall study asymptotic behavior of solutions to (17) with appropriate initial data using spectral parameters of a leafwise Schrödinger operator

$$\mathcal{H}^\top = -\Delta^\top - \beta^\top.$$

The least eigenvalue λ_0^\top of \mathcal{H}^\top is simple and obeys the inequalities

$$\lambda_0^\top \in [-\max_F \beta^\top, -\min_F \beta^\top],$$

its eigenfunction e_0 (called the ground state) may be chosen positive, see Sect. 2. By (2) and results in Sect. 3, the leafwise constant λ_0^\top and e_0 are smooth on M .

Assume $h \neq 0$ and $\Phi < n\lambda_0^\top$ and consider the functions (compare with Sect. 2),

$$\begin{aligned} \phi_-^\top(y) &= -(n\lambda_0^\top - \Phi)y + \min_F (\|h\|^2 e_0^{-2})y^{-1} - \max_F (\|T\|^2 e_0^{-4})y^{-3}, \\ \phi_+^\top(y) &= -(n\lambda_0^\top - \Phi)y + \max_F (\|h\|^2 e_0^{-2})y^{-1} - \min_F (\|T\|^2 e_0^{-4})y^{-3}. \end{aligned} \quad (18)$$

If the discriminant $D = \min_F (\|h\|^4 e_0^{-4}) - 4(n\lambda_0^\top - \Phi) \max_F (\|T\|^2 e_0^{-4}) > 0$, each of (18) has four real roots (two of them are positive). Their maximal (positive) roots

$$\begin{aligned} y_-^\top &= \left(\frac{\min_F (\|h\|^2 e_0^{-2}) + (\min_F (\|h\|^4 e_0^{-4}) - 4(n\lambda_0^\top - \Phi) \max_F (\|T\|^2 e_0^{-4}))^{1/2}}{2(n\lambda_0^\top - \Phi)} \right)^{\frac{1}{2}}, \\ y_+^\top &= \left(\frac{\max_F (\|h\|^2 e_0^{-2}) + (\max_F (\|h\|^4 e_0^{-4}) - 4(n\lambda_0^\top - \Phi) \min_F (\|T\|^2 e_0^{-4}))^{1/2}}{2(n\lambda_0^\top - \Phi)} \right)^{\frac{1}{2}}, \end{aligned}$$

obey the inequalities $y_-^\top < y_3^\top < y_+^\top$, where y_3^\top is the maximal root of $(\phi_-^\top)'(y)$,

$$y_3^\top = \left(\frac{(\min_F (\|h\|^4 e_0^{-4}) + 12(n\lambda_0^\top - \Phi) \max_F (\|T\|^2 e_0^{-4}))^{1/2} - \min_F (\|h\|^2 e_0^{-2})}{2(n\lambda_0^\top - \Phi)} \right)^{\frac{1}{2}}.$$

For a positive function $f \in C(F)$ define $\delta_f := (\min_F f)/(\max_F f) \in (0, 1]$.

Proposition 3. *Let \mathcal{F} be a harmonic and nowhere totally geodesic foliation on a Riemannian manifold (M, g) with condition (2) and $H^\perp = 0$. Then for any leafwise constant $\Phi \in C^\infty(M)$ obeying the inequalities (along any leaf F)*

$$n\lambda_0^\top - \delta_{e_0}^{-4} \frac{\min_F \|h\|^4}{4 \max_F \|T\|^2} < \Phi < n\lambda_0^\top, \quad (19)$$

there exists a unique u_* in the set $\{\tilde{u} \in C^\infty(M) : \tilde{u} > e_0 y_3^\top\}$ such that the mixed scalar curvature of $\tilde{g} = g_{\mathcal{F}} + u_*^2 g^\perp$ is Φ . Moreover, $y_-^\top \leq u_*/e_0 \leq y_+^\top$ and $u_* = \lim_{t \rightarrow \infty} u(\cdot, t)$, where u solves (17) with $\tilde{S}_{\text{mix}} = \Phi$, does not depend on the value $u(\cdot, 0) = u_0 > e_0 y_3^\top$.

Proof. By Theorem 4 (in Sect. 3), the leafwise constant λ_0^\top (the least eigenfunction of \mathcal{H}^\top) and its leafwise eigenvector e_0 are smooth, i.e., they belong to $C^\infty(M)$. If M is closed there then exist many Φ 's obeying (19), e.g. $n\lambda_0^\top - \varepsilon$ for small enough $\varepsilon > 0$.

By conditions, any leaf F_0 has an open neighborhood diffeomorphic to $F \times \mathbb{R}^n$ and $F_0 = F \times \{0\}$. Since $F_q = F \times \{q\}$ are compact minimal submanifolds, their volume form $d\text{vol}_F = |g|_F|^{1/2} dx$ does not depend on $q \in \mathbb{R}^n$, see [16]. Thus, vector bundles $\{L_2(F_q)\}_{q \in \mathbb{R}^n}$ and $\{H^k(F_q)\}_{q \in \mathbb{R}^n}$ coincide with products $L_2 \times \mathbb{R}^n$ and $H^k \times \mathbb{R}^n$.

Let Φ obey (19) and let $u_0 > e_0 y_3^\top$ hold. We shall use the notation

$$\beta = \beta^\top + \Phi/n, \quad \lambda_0 = \lambda_0^\top - \Phi/n, \quad \Psi_1 = \|h\|^2/n, \quad \Psi_2 = \|T\|^2/n.$$

Then (17) with $\tilde{S}_{\text{mix}} = \Phi$ becomes (21), while (19) follows from

$$n\lambda_0^\top - (1/4) \min_F (\|h\|^4 e_0^{-4}) / \max_F (\|T\|^2 e_0^{-4}) < \Phi < n\lambda_0^\top,$$

which becomes (26). Hence, the claim follows from Theorem 3 (in Sect. 2). \square

Corollary 2. *Let \mathcal{F} be a harmonic and nowhere totally geodesic foliation of a Riemannian manifold (M, g) with condition (2), integrable normal subbundle \mathcal{D}^\perp and $H^\perp = 0$. Then for any leafwise constant $\Phi \in C^\infty(M)$ such that $\Phi < n\lambda_0^\top$ there exists a unique positive function $u_* \in C^\infty(M)$ such that (along any leaf F)*

$$(n\lambda_0^\top - \Phi)^{-1} \min_F (\|h\|^2 e_0^{-2}) \leq u_*/e_0 \leq (n\lambda_0^\top - \Phi)^{-1} \max_F (\|h\|^2 e_0^{-2}),$$

and the mixed scalar curvature of the metric $\tilde{g} = g_{\mathcal{F}} + u_*^2 g^\perp$ is Φ .

Proof. This is similar to the proof of Proposition 3. Since $\Psi_2 \equiv 0$ and $\lambda_0 > 0$, each of $\phi_- = -\lambda_0 + \Psi_1^- y^{-1}$ and $\phi_+ = -\lambda_0 + \Psi_1^+ y^{-1}$ has one positive root $y_1^- = (\Psi_1^-/\lambda_0)^{1/2}$ and $y_1^+ = (\Psi_1^+/\lambda_0)^{1/2}$, see also Example 3(c). \square

Proof of Theorem 1. By Proposition 1, there exists a metric g_1 , \mathcal{D}^\perp -conformal to g , for which $H^\perp = 0$ (the mean curvature of \mathcal{D}^\perp). By Lemma 2, the equality $H = 0$ is preserved for g_1 . By Proposition 3, there exists a metric \tilde{g} , \mathcal{D}^\perp -conformal to g_1 , for which \tilde{S}_{mix} is leafwise constant; moreover, $H = 0$ holds. \square

Proof of Theorem 2. By Corollary 2, there is a metric g_1 that is \mathcal{D}^\perp -conformal to g , for which $H^\perp = 0$. By Lemma 2, $h = 0$ is preserved for g_1 . Since $T = 0$, equation (5) reads as the eigenproblem $\mathcal{H}(u) = \frac{1}{n} \tilde{S}_{\text{mix}} u$, where $\mathcal{H} = -\Delta^\top - \beta$ is a leafwise Schrödinger operator on (M, g_1) with potential $\beta = -\frac{1}{n} S_{\text{mix}}(g_1)$. Let $e_0 > 0$ be the ground state of \mathcal{H} with the least eigenvalue λ_0^\top (leafwise constant). Thus, the metric $\tilde{g} = g_{\mathcal{F}} + e_0^2 g_1^\perp$ has $\tilde{S}_{\text{mix}} = n\lambda_0^\top$; moreover, the equality $h = 0$ is preserved for \tilde{g} . \square

2 Results for the nonlinear heat equation

Let (F, g) be a smooth closed p -dimensional Riemannian manifold (e.g., a leaf of a compact foliation) with the Riemannian distance $d(x, y)$. Functional spaces over F will be denoted without writing (F) , e.g., L_2 instead of $L_2(F)$. Let H^k be the Hilbert space of Sobolev real functions of order k on F with the inner product $(\cdot, \cdot)_k$ and the norm $\|\cdot\|_k$. In particular, $H^0 = L_2$ with the product $(\cdot, \cdot)_0$ and the norm $\|\cdot\|_0$. Denote by $\|\cdot\|_{C^k}$ the norm in the Banach space C^k for $1 \leq k < \infty$, and $\|\cdot\|_C$ for $k = 0$. In local coordinates (x_1, \dots, x_p) on F , we have $\|f\|_{C^k} = \max_F \max_{|m| \leq k} |d^m f|$, where $m \geq 0$ is the multi-index of order $|m| = \sum_i m_i$ and d^m is the partial derivative (in fact, a finite atlas of F must be considered).

Proposition 4 (Scalar maximum principle, see [7, Theorem 4.4]). *Let X_t and g_t be smooth families of vector fields and metrics on a closed Riemannian manifold F , and $f \in C^\infty(\mathbb{R} \times [0, T])$. Suppose that $u : F \times [0, T] \rightarrow \mathbb{R}$ is a C^∞ solution to*

$$\partial_t u \geq \Delta_t u - X_t(u) + f(u, t),$$

and let $y : [0, T] \rightarrow \mathbb{R}$ solve the Cauchy's problem for ODEs: $y' = f(y(t), t)$, $y(0) = y_0$. If $u(\cdot, 0) \geq y_0$, then $u(\cdot, t) \geq y(t)$ for all $t \in [0, T]$.

2.1. The nonlinear heat equation. We are looking for stable solutions of the elliptic equation, see (5) with $H^\perp = 0$,

$$-\Delta u - \beta u = \Psi_1(x) u^{-1} - \Psi_2(x) u^{-3}, \quad (20)$$

where $\Psi_1 > 0$, $\Psi_2 \geq 0$ and β are smooth functions on F . To study (20), we shall look for attractors of the Cauchy's problem for the nonlinear heat equation,

$$\partial_t u = \Delta u + \beta u + \Psi_1(x) u^{-1} - \Psi_2(x) u^{-3}, \quad u(x, 0) = u_0(x) > 0. \quad (21)$$

Let $\mathcal{C}_t = F \times [0, t]$ be cylinder with the base F . By [2, Theorem 4.51], (21) has a unique smooth solution in \mathcal{C}_{t_0} for some $t_0 > 0$. Let S_t be a map which relates to each initial value $u_0 \in C$ the value of this solution at $t \in [0, t_0]$. Since rhs of (21) does not depend explicitly on t , the family $\{S_t\}$ has the semigroup property, and it is a semigroup (i.e., $t_0 = \infty$) when (21) has a global solution for any $u_0(x) \in C$.

Let $\mathcal{H} := -\Delta - \beta$ be a Schrödinger operator with domain in H^2 , and $\sigma(\mathcal{H})$ the spectrum. One can add a real constant to β such that \mathcal{H} becomes invertible in L_2 (e.g. $\lambda_0 > 0$) and \mathcal{H}^{-1} is bounded in L_2 .

Theorem (Elliptic regularity, see [2]). *If $0 \notin \sigma(\mathcal{H})$, then for any integer $k \geq 0$ we have $\mathcal{H}^{-1} : H^k \rightarrow H^{k+2}$.*

By the Elliptic regularity Theorem with $k = 0$, we have $\mathcal{H}^{-1} : L_2 \rightarrow H^2$, and the embedding of H^2 into L_2 is continuous and compact, see [2]. Hence, the operator $\mathcal{H}^{-1} : L_2 \rightarrow L_2$ is compact. Thus, the spectrum $\sigma(\mathcal{H})$ is discrete, i.e., consists of an infinite sequence of real eigenvalues $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_j \leq \dots$ with finite multiplicities, bounded from below

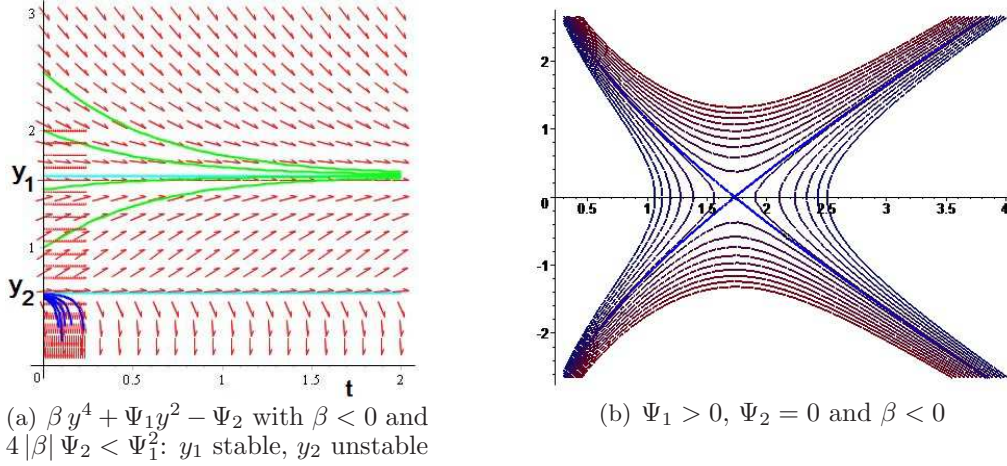


Figure 1: Example 3: the nonlinear heat equation

and $\lim_{j \rightarrow \infty} \lambda_j = \infty$. One may fix in L_2 an orthonormal basis of eigenfunctions $\{e_j\}$, i.e., $\mathcal{H}(e_j) = \lambda_j e_j$. Since the eigenvalue λ_0 is simple, its eigenfunction $e_0(x)$ (called the *ground state*) can be chosen positive, see [19, Prop. 3].

The following examples show us that (21) may have

- (i) stationary (i.e., t -independent) solutions on a closed manifold F ;
- (ii) attractors (i.e., asymptotically stable stationary solutions) when $\beta < 0$.

Example 3. Let β and $\Psi_1 > 0$, $\Psi_2 \geq 0$ be real constants. Then (21) is the Cauchy's problem for ODE

$$y' = f(y), \quad y(0) = y_0 > 0, \quad f(u) := \beta u + \Psi_1 u^{-1} - \Psi_2 u^{-3}. \quad (22)$$

(a) Let $\beta < 0$ and $\Psi_2 > 0$. Positive stationary (i.e., constant) solutions of (22) are the roots of a biquadratic equation $y^3 f(y) = 0$. If $4|\beta|\Psi_2 < \Psi_1^2$, then we have two positive solutions $y_{1,2} = \left(\frac{\Psi_1 \pm (\Psi_1^2 - 4|\beta|\Psi_2)^{1/2}}{2|\beta|} \right)^{1/2}$ and $y_1 > y_2$. The linearization of (22) at the point y_k ($k = 1, 2$) is $v' = f'(y_k)v$, where $f'(y_k) = -|\beta|(y^{-3}(y^2 - y_1^2)(y^2 - y_2^2))'|_{y=y_k}$. Hence, $f'(y_1) < 0$ and $f'(y_2) > 0$, and y_1 is asymptotically stable, but y_2 is unstable. If $4|\beta|\Psi_2 = \Psi_1^2$, then (22) has one positive stationary solution, see Fig. 1(a), and has no stationary solutions if $4|\beta|\Psi_2 > \Psi_1^2$.

(b) Let $\beta > 0$ and $\Psi_2 > 0$. Then the biquadratic equation $y^3 f(y) = 0$ has one positive root $y_1 = \left(\frac{-\Psi_1 + (\Psi_1^2 + 4\beta\Psi_2)^{1/2}}{2\beta} \right)^{1/2}$. We find

$$f'(y_1) = \beta(y^{-3}(y^2 - y_1^2)(y^2 + \Psi_2/(\beta y_1^2)))'|_{y=y_1} > 0;$$

hence, y_1 is unstable. One may also show that in the case $\beta = 0$, (22) has a unique positive stationary solution, which is unstable.

(c) Let $\Psi_2 = 0$ and $\Psi_1 > 0$. Then $f(y) = \beta y + \Psi_1 y^{-1}$. If $\beta \geq 0$, then there are no positive stationary solutions. If $\beta < 0$, then f has one positive root $y_1 = (\Psi_1/|\beta|)^{1/2}$. Since $f'(y_1) = -|\beta|(y^{-1}(y - y_1)(y + y_1))'|_{y=y_1} < 0$, the solution y_1 is an attractor.

Example 4. Let F be a circle S^1 of length l . Then (21) is the Cauchy's problem

$$u_t = u_{xx} + f(u), \quad u(x, 0) = u_0(x) > 0 \quad (x \in S^1, t \geq 0). \quad (23)$$

The stationary equation with $u(x)$ for (23) has the form

$$u'' + f(u) = 0, \quad u(0) = u(l), \quad u'(0) = u'(l), \quad l > 0. \quad (24)$$

Rewrite (24) as the dynamical system

$$u' = v, \quad v' = -f(u) \quad (u > 0). \quad (25)$$

Periodic solutions of (24) correspond to solutions of (25) with the same period. The system (25) is Hamiltonian, since $\partial_u v = \partial_v f(u)$, its Hamiltonian $H(u, v)$ (the first integral) solves $\partial_u H(u, v) = f(u)$, $\partial_v H(u, v) = v$. Then $H(u, v) = \frac{1}{2}(v^2 + \beta u^2) + \Psi_1 \ln u + \frac{1}{2}\Psi_2 u^{-2}$. The trajectories of (25) belong to level lines of $H(u, v)$. Consider the cases.

(a) Assume $\beta < 0$. Then (25) has two fixed points: $(y_i, 0)$ ($i = 1, 2$) with $y_1 > y_2$. To clear up the type of fixed points, we linearize (25) at $(y_i, 0)$,

$$\vec{\eta}' = A_i \vec{\eta}, \quad A_i = \begin{pmatrix} 0 & 1 \\ -f'(y_i) & 0 \end{pmatrix}.$$

Since $f'(y_1) < 0$ and $f'(y_2) > 0$, the point $(y_1, 0)$ is a “saddle” and $(y_2, 0)$ is a “center”. The separatrix is $H(u, v) = H(y_1, 0)$, i.e., see Fig. 2(a),

$$v^2 = |\beta|(u^2 - y_1^2) - 2\Psi_1 \ln(u/y_1) - \Psi_2(u^{-2} - y_1^{-2}).$$

The separatrix divides the half-plane $u > 0$ into three simply connected areas. Then $(y_2, 0)$ is a unique minimum point of H in $D = \{(u, v) : H(u, v) < H(y_1, 0), 0 < u < y_1\}$. The phase portrait of (25) in D consists of the cycles surrounding the fixed point $(y_2, 0)$, all correspond to non-constant solutions of (24) with various l . Other two areas do not contain cycles, since they have no fixed points.

(b) Assume $\beta \geq 0$. Then (25) has one fixed point $(y_1, 0)$ and $f'(y_1) > 0$. Hence, $(y_1, 0)$ is a “center”. Since $(y_1, 0)$ is a unique minimum of $H(u, v)$ in the semiplane $u > 0$, the phase portrait of (25) consists of the cycles surrounding the fixed point $(y_1, 0)$, all correspond to non-constant solutions of (24) with various l , see Fig. 2(b).

For $\Psi_2 = 0$ and $\Psi_1 > 0$, the Hamiltonian of (25) is $H(u, v) = \frac{1}{2}(v^2 + \beta u^2) + \Psi_1 \ln u$. Solving $H(u, v) = C$ with respect to v and substituting to (25)₁, we get $u' = \sqrt{-\beta u^2 - 2\Psi_1 \ln u + 2C}$. If $\beta \geq 0$, then (25) has no cycles (since it has no fixed points); hence, (24) has no solutions. If $\beta < 0$, then the separatrix $H(u, v) = H(u_*, 0)$, see Example 3(c), is $v^2 = |\beta|(u^2 - u_*^2) - 2\Psi_1 \ln(u/u_*)$, (25) has a unique fixed point $(u_*, 0)$ which is a “saddle”. The separatrix divides the half-plane $u > 0$ into four simply connected areas with these lines, see Fig. 1(b). Since each of these areas has no fixed points of (25), the system has no cycles. Hence, u_* is a unique solution of (24).

(c) Consider (24) for $\Psi_1 = 0$, $\Psi_2 > 0$ and $l = 2\pi$. Set $p = u'$ and represent $p = p(u)$ as a function of u . Then $u'' = dp/du$ and

$$(p^2)' = -2\beta u + 2\Psi_2 u^{-3} \Rightarrow (u')^2 = C_1 - \beta u^2 - \Psi_2 u^{-2}.$$

After separation of variables and integration, we obtain

$$u = \begin{cases} \sqrt{\frac{C_1}{2\beta} + \frac{1}{2\beta} \sqrt{C_1^2 - 4\beta\Psi_2 \sin(2\sqrt{\beta}(x+C_2))}}, & (C_1^2 \geq 4\beta\Psi_2) \text{ for } \beta > 0, \\ \sqrt{-\frac{C_1}{2|\beta|} + \frac{1}{2|\beta|} \sqrt{C_1^2 + 4|\beta|\Psi_2 \cosh(2\sqrt{|\beta|}(x+C_2))}}, & \text{for } \beta < 0, \\ \sqrt{\Psi_2/C_1 + C_1(x+C_2)^2} & \text{for } \beta = 0. \end{cases}$$

Hence, for $\beta \leq 0$, (24) has no positive solutions, while for $\beta > 0$ the solution is 2π -periodic and positive only if

- $\beta \neq n^2/4$ ($n \in \mathbb{N}$) and $C_1 = 2(\beta\Psi_2)^{1/2}$; a solution $u_* = (\Psi_2/\beta)^{1/4}$ is unique, or
- $\beta = n^2/4$ ($n \in \mathbb{N}$); the set of solutions forms a two-dimensional manifold

$$n u_0(C_1, C_2) = (2C_1 + 2(C_1^2 - n^2\Psi_2)^{1/2} \sin(n(x+C_2)))^{1/2}.$$

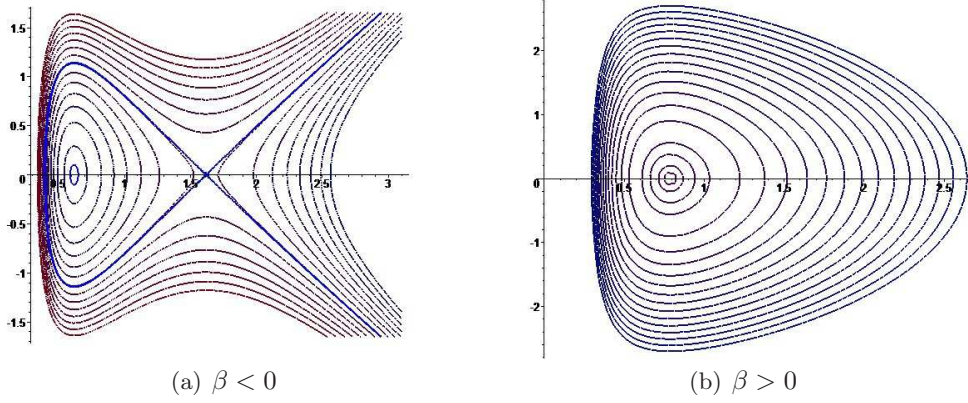


Figure 2: Example 4.

2.2. Attractor of the nonlinear heat equation. Denote by $\Psi_i^+ = \max_F(\Psi_i e_0^{-2i})$ and $\Psi_i^- = \min_F(\Psi_i e_0^{-2i})$ for $i = 1, 2$. Let $\Psi_2^+ > 0$ (the case of $\Psi_2^+ = 0$ is similar) and

$$0 < \lambda_0 < (\Psi_1^-)^2 / (4 \Psi_2^+). \quad (26)$$

Each of the two functions of variable $y > 0$,

$$\phi_+(y) = -\lambda_0 y + \Psi_1^+ y^{-1} - \Psi_2^- y^{-3}, \quad \phi_-(y) = -\lambda_0 y + \Psi_1^- y^{-1} - \Psi_2^+ y^{-3}, \quad (27)$$

has four real roots, two of which, $y_2^+ < y_1^+$ and $y_2^- < y_1^-$, are positive. Since $\phi_-(y) \leq \phi_+(y)$ for $y > 0$, we also have $y_1^- \leq y_1^+$. Denote by

$$y_3^- = \left(\frac{-\Psi_1^- + ((\Psi_1^-)^2 + 12 \Psi_2^+ \lambda_0)^{1/2}}{2 \lambda_0} \right)^{1/2} \quad (28)$$

a unique positive root of $\phi'_-(y)$. Clearly, $y_3^- \in (y_2^-, y_1^-)$. Notice that $\phi_-(y) > 0$ for $y \in (y_2^-, y_1^-)$ and $\phi_-(y) < 0$ for $y \in (0, \infty) \setminus [y_2^-, y_1^-]$; moreover, $\phi_-(y)$ increases in $(0, y_3^-)$ and decreases in (y_3^-, ∞) . The line $z = -\lambda_0 y$ is asymptotic for the graph of $\phi_-(y)$ when $y \rightarrow \infty$, and $\lim_{y \downarrow 0} \phi_-(y) = -\infty$. Next, $\phi'_-(y)$ decreases in $(0, y_4^-)$ and increases in (y_4^-, ∞) , where $y_4^- := (6 \Psi_2^+ / \Psi_1^-)^{1/2} > y_3^-$, and $\lim_{y \rightarrow \infty} \phi'_-(y) = -\lambda_0$, see Fig. 3. Hence,

$$\mu^-(\sigma) := -\sup_{y \geq y_1^- - \sigma} \phi'_-(y) = \min\{|\phi'_-(y_1^- - \sigma)|, \lambda_0\} > 0 \quad (29)$$

for $\sigma \in (0, y_1^- - y_3^-)$. Similar properties have y_3^+, y_4^+ and $\mu^+(\sigma)$ defined for $\phi'_+(y)$.

Lemma 3. *Let $y(t)$ be a solution of Cauchy's problem*

$$y' = \phi_-(y), \quad y(0) = y_0^- > 0. \quad (30)$$

- (i) *If $y_0^- > y_2^-$ then $\lim_{t \rightarrow \infty} y(t) = y_1^-$. Furthermore, if $y_0^- \in (y_2^-, y_1^-)$ then $y(t)$ is increasing and if $y_0^- > y_1^-$ then $y(t)$ is decreasing.*
- (ii) *If $y_0^- \geq y_1^- - \varepsilon$ for some $\varepsilon \in (0, y_1^- - y_3^-)$ then*

$$|y(t) - y_1^-| \leq |y_0^- - y_1^-| e^{-\mu^-(\varepsilon)t}. \quad (31)$$

Similar claims are valid for Cauchy's problem $y' = \phi_+(y)$, $y(0) = y_0^+ > 0$.

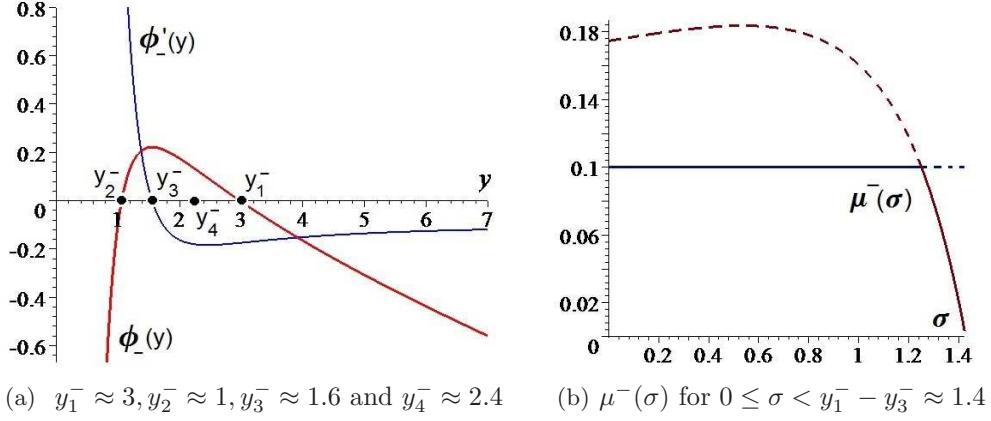


Figure 3: Graphs of ϕ_- , ϕ'_- and μ^- for $\Psi_1 = \Psi_2 = 1$ and $\lambda_0 = 0.1$.

Proof. (i) Assume that $y_0^- \in (y_2^-, y_1^-)$. Since $\phi_-(y)$ is positive in (y_2^-, y_1^-) , $y(t)$ is increasing. The graph of $y(t)$ cannot intersect the graph of the stationary solution y_1^- ; hence, the solution $y(t)$ exists and is continuous on the whole $[0, \infty)$, and it is bounded there. There exists $\lim_{t \rightarrow \infty} y(t)$, which coincides with y_1^- , since y_1^- is a unique solution of $\phi(y) = 0$ in (y_2^-, ∞) . The case $y_0^- > y_1^-$ is treated similarly. Notice that if $y_0^- \in (y_2^-, y_1^-)$ then $y(t)$ is increasing, and if $y_0^- > y_1^-$ then $y(t)$ is decreasing.

(ii) For $y_0^- \geq y_1^- - \varepsilon$, where $\varepsilon \in (0, y_1^- - y_3^-)$, denote $z(t) = y_1^- - y(t)$. We obtain from (30), using definition of $\mu^-(\varepsilon)$ and the fact that $\phi_-(y_1^-) = 0$,

$$(z^2)' = 2z z' = 2z^2 \int_0^1 \phi'_-(y + \tau z) d\tau \leq -2\mu^-(\varepsilon) z^2.$$

This differential inequality implies (31). The case $y_0^- > y_1^-$ is treated similarly. \square

Under assumption (26), define nonempty sets $\mathcal{U}_2^{\varepsilon, \eta} \subset \mathcal{U}_1^\varepsilon$, closed in C , with $\varepsilon \in (0, y_1^- - y_3^-)$ and $\eta > 0$ by

$$\mathcal{U}_1^\varepsilon = \{u_0 \in C : u_0/e_0 \geq y_1^- - \varepsilon\}, \quad \mathcal{U}_2^{\varepsilon, \eta} = \{u_0 \in C : y_1^- - \varepsilon \leq u_0/e_0 \leq y_1^+ + \eta\}.$$

Then, $\mathcal{U}_1^\varepsilon \subset \mathcal{U}_1$, where $\mathcal{U}_1 = \{u_0 \in C : u_0/e_0 > y_3^-\}$ is open in C .

Proposition 5. *Let (26) hold. Then Cauchy's problem (21) with $u_0 \in \mathcal{U}_1^\varepsilon$ for some $\varepsilon \in (0, y_1^- - y_3^-)$, admits a unique global solution. Furthermore, the sets $\mathcal{U}_1^\varepsilon$ and $\mathcal{U}_2^{\varepsilon, \eta}$ ($\eta > 0$) are invariant for the semigroup of operators corresponding to (21)₁.*

Proof. Let $u(\cdot, t)$ ($t \geq 0$) solve (21) with $u_0 \in \mathcal{U}_1^\varepsilon$ for some $\varepsilon \in (0, y_1^- - y_3^-)$. Substituting $u = e_0 w$ and using $\Delta e_0 + \beta e_0 = -\lambda_0 e_0$, yields the Cauchy's problem

$$\partial_t w = \Delta w + \langle 2\nabla \log e_0, \nabla w \rangle + f(w, \cdot), \quad w(\cdot, 0) = u_0/e_0 \geq y_1^- - \varepsilon, \quad (32)$$

for $w(x, t)$, where

$$f(w, \cdot) = -\lambda_0 w + (\Psi_1 e_0^{-2}) w^{-1} - (\Psi_2 e_0^{-4}) w^{-3}. \quad (33)$$

From (32) and (27) we obtain the differential inequalities

$$\phi_-(w) \leq \partial_t w - \Delta w - \langle 2\nabla \log e_0, \nabla w \rangle \leq \phi_+(w). \quad (34)$$

By Proposition 4, applied to the left inequality of (34), and Lemma 3, in the maximal domain D_M of the existence of the solution $w(x, t)$ of (32), we obtain the inequality

$$w(\cdot, t) \geq y_1^- - \varepsilon e^{-\mu^-(\varepsilon)t} \geq y_1^- - \varepsilon > 0,$$

which implies that $w(x, t)$ cannot “blowdown” to zero. Since $\phi_+(w) \leq \Psi_1^+ w^{-1}$, from the right inequality of (34), applying Proposition 4, we obtain in D_M

$$w(\cdot, t) \leq w_+(t) = (((u_0^+)^2 - \Psi_1^+/\lambda_0) e^{-2\lambda_0 t} + \Psi_1^+/\lambda_0)^{1/2},$$

where $w_+(t)$ solves the Cauchy’s problem for ODE

$$dw_+/dt + \lambda_0 w_+ = \Psi_1^+ w_+^{-1}, \quad w_+(0) = u_0^+ := \max_F (u_0/e_0).$$

By the above, the solution $u(x, t)$ of (21) exists for all $(x, t) \in \mathcal{C}_\infty$, and the set $\mathcal{U}_1^\varepsilon$ is invariant for the semigroup of operators $\mathcal{S}_t : u_0 \mapsto u(\cdot, t)$ ($t \geq 0$) in $\mathcal{C}_\infty = F \times [0, \infty)$, corresponding to $(21)_1$. Assuming $u_0 \in \mathcal{U}_1^{\varepsilon, \eta}$ and applying again Proposition 4 and Lemma 3 to the right inequality of (34), we get

$$w(\cdot, t) \leq y_1^+ + \eta e^{-\mu^+(\sigma)t}, \quad \sigma \in (0, y_1^+ - y_3^+).$$

Thus, $u(\cdot, t) \in \mathcal{U}_1^{\varepsilon, \eta}$ ($t > 0$). Hence, also the set $\mathcal{U}_2^{\varepsilon, \eta}$ is invariant for all \mathcal{S}_t . \square

Theorem 3. (i) If (26) holds then (20) admits in \mathcal{U}_1 a unique solution u_* (on F), which is smooth; moreover, $u_* = \lim_{t \rightarrow \infty} u(\cdot, t)$, where u solves (21) with $u_0 \in \mathcal{U}_1$, and $y_1^- \leq u_*/e_0 \leq y_1^+$. Furthermore, for any $\varepsilon \in (0, y_1^- - y_3^-)$, the set $\mathcal{U}_1^\varepsilon$ is attracted by $(21)_1$ exponentially fast to the point u_* in C -norm:

$$\|u(\cdot, t) - u_*\|_C \leq \delta_{e_0}^{-1} e^{-\mu^-(\varepsilon)t} \|u_0 - u_*\|_C \quad (t > 0, u_0 \in \mathcal{U}_1^\varepsilon). \quad (35)$$

(ii) If β, Ψ_1, Ψ_2 are smooth functions on the product $F \times \mathbb{R}^n$ with a smooth leafwise metric $g(\cdot, q)$ and (26) holds for any leaf $F \times \{q\}$ ($q \in \mathbb{R}^n$) then the leafwise solution u_* of (20) is smooth on $F \times \mathbb{R}^n$.

Proof. (i) By Proposition 5, the set $\mathcal{U}_1^\varepsilon$ is invariant for the semigroup of operators $\mathcal{S}_t : u_0 \rightarrow u(\cdot, t)$ ($t \geq 0$) corresponding to $(21)_1$, i.e., $\mathcal{S}_t(\mathcal{U}_1^\varepsilon) \subseteq \mathcal{U}_1^\varepsilon$ for $t \geq 0$. Take $u_i^0 \in \mathcal{U}_1^\varepsilon$ ($i = 1, 2$) and denote by

$$u_i(\cdot, t) = \mathcal{S}_t(u_i^0), \quad w_i(\cdot, t) = u_i(\cdot, t)/e_0, \quad w_i^0 = u_i^0/e_0.$$

Using (32) and the equalities

$$2 \bar{w} \Delta \bar{w} = \Delta(\bar{w}^2) - 2 \|\nabla \bar{w}\|^2, \quad \nabla(\bar{w}^2) = 2 \bar{w} \nabla \bar{w}$$

with $\bar{w} = w_2 - w_1$, we obtain

$$\begin{aligned} \partial_t((w_2 - w_1)^2) &= 2(w_2 - w_1) \partial_t(w_2 - w_1) \leq \Delta((w_2 - w_1)^2) \\ &+ \langle 2 \nabla \log e_0, \nabla(w_2 - w_1)^2 \rangle + 2(f(w_2, \cdot) - f(w_1, \cdot))(w_2 - w_1). \end{aligned}$$

We estimate the last term, using $w_i \geq y_1^- - \varepsilon > y_3^-$ ($i = 1, 2$), (29) and (33),

$$\begin{aligned} &(f(w_2, \cdot) - f(w_1, \cdot))(w_2 - w_1) \\ &= (w_2 - w_1)^2 \int_0^1 \partial_w f(w_1 + \tau(w_2 - w_1), \cdot) d\tau \leq -\mu^-(\varepsilon)(w_2 - w_1)^2. \end{aligned}$$

Thus, the function $v = (w_2 - w_1)^2$ satisfies the differential inequality

$$\partial_t v \leq \Delta v + \langle 2 \nabla \log e_0, \nabla v \rangle - 2\mu^-(\varepsilon)v.$$

By Proposition 4, $v(\cdot, t) \leq v_+(t)$, where $v_+(t)$ solves the Cauchy's problem for ODE:

$$v'_+ = -2\mu^-(\varepsilon) v_+(t), \quad v_+(0) = \|w_2^0 - w_1^0\|_C^2.$$

Thus,

$$\begin{aligned} \|\mathcal{S}_t(u_2^0) - \mathcal{S}_t(u_1^0)\|_C &\leq \|w_2(\cdot, t) - w_1(\cdot, t)\|_C \cdot \max_F e_0 \\ &\leq e^{-\mu^-(\varepsilon)t} \|w_2^0 - w_1^0\|_C \cdot \max_F e_0 \leq \delta_{e_0}^{-1} e^{-\mu^-(\varepsilon)t} \|u_2^0 - u_1^0\|_C, \end{aligned}$$

i.e., the operators \mathcal{S}_t ($t \geq 0$) corresponding to (21) satisfy in $\mathcal{U}_1^\varepsilon$ the Lipschitz condition with respect to C -norm with the Lipschitz constant $\delta_{e_0}^{-1} e^{-\mu^-(\varepsilon)t}$.

By Proposition 5, for any $t \geq 0$ the operator \mathcal{S}_t for (21) maps the set $\mathcal{U}_1^\varepsilon$, which is closed in C , into itself, and for $t > \frac{1}{\mu^-(\varepsilon)} \ln \delta_{e_0}^{-1}$ it is a contraction there. Since all operators \mathcal{S}_t commute one with another, they have a unique common fixed point u_* in $\mathcal{U}_1^\varepsilon$. Since $\varepsilon \in (0, y_1^- - y_3^-)$ is arbitrary, u_* is a unique common fixed point of all \mathcal{S}_t in the set \mathcal{U}_1 . For any $u_0 \in \mathcal{U}_1^\varepsilon$ and $t \geq 0$, (35) holds. Thus, $u_* \in C$ is a generalized solution of (20). By the Elliptic Regularity Theorem, $u_* \in C^\infty$ and it is a classical solution. By Proposition 5, $\mathcal{U}_2^{\varepsilon, \eta} \subset \mathcal{U}_1^\varepsilon$ is also \mathcal{S}_t -invariant, hence $u_* \in \mathcal{U}_2^{\varepsilon, \eta}$. Since $\varepsilon \in (0, y_1^- - y_3^-)$ and $\eta > 0$ are arbitrary, we get $y_1^- \leq u_*/e_0 \leq y_1^+$.

Notice that if the functions Ψ_1 and Ψ_2 are constant then $\phi_+ = \phi_-$, see (27); in this case, $u_*/e_0 = y_1^+ = y_1^-$ is constant, too.

(ii) Let $e_0(x, q) > 0$ be the normalized eigenfunction for the minimal eigenvalue $\lambda_0(q)$ of the operator $\mathcal{H}_q = -\Delta - \beta(x, q)$. By Theorem 4 (in Sect. 3), $\lambda_0 \in C^\infty(\mathbb{R}^n)$ and $e_0 \in C^\infty(F \times \mathbb{R}^n)$, hence y_3^- , defined by (28), smoothly depends on q . As we have proved in (i), for any $q \in \mathbb{R}^n$ the stationary equation, see also (20),

$$\Delta_q u + f(u, x, q) = 0, \tag{36}$$

with $f(u, x, q) = \beta(x, q)u + \Psi_1(x, q)u^{-1} - \Psi_2(x, q)u^{-3}$ has a unique solution $u_*(x, q)$ in the open set $\mathcal{U}_1(q) = \{u_0 \in C(F \times \mathbb{R}^n) : u_0/e_0(\cdot, q) > y_3^-(q)\}$.

Since $y_3^-(q)$ and $e_0(x, q)$ are continuous, there exist open neighborhoods $U_* \subseteq C^{k+2, \alpha}$ of $u_*(x, 0)$ and $V_0 \subset \mathbb{R}^n$ of 0 such that

$$U_* \subseteq \mathcal{U}_1(q) \quad \forall q \in V_0. \tag{37}$$

We claim that all eigenvalues of the linear operator $\mathcal{H}_* = -\Delta_0 - \partial_u f(u_*(x, 0), x, 0)$, acting in L_2 with the domain H^2 , are positive. To show this, observe that $y_1^-(0) \leq u_*(\cdot, 0)/e_0(\cdot, 0) \leq y_1^+(0)$. Let $\tilde{u}(x, t)$ be a solution of Cauchy's problem for the evolution equation

$$\partial_t \tilde{u} = -\mathcal{H}_*(\tilde{u}), \quad \tilde{u}(x, 0) = \tilde{u}_0(x) \in C. \tag{38}$$

Using the same arguments as in the proof of (i), we obtain that the function $v(x, t) = \tilde{u}^2(x, t)e_0^{-2}(x, 0)$ obeys the differential inequality with $\mu_0^- = \min\{|\phi'_-(y_1^-)|, \lambda_0\} > 0$:

$$\partial_t v \leq \Delta_0 v + \langle 2 \nabla \log e_0(\cdot, 0), \nabla v \rangle - 2\mu_0^- v.$$

By Proposition 4, $v(\cdot, t) \leq v_+(t)$, where $v_+(t)$ solves the Cauchy's problem for ODE

$$v'_+ = -2\mu_0^- v_+, \quad v_+(0) = \|\tilde{u}_0/e_0(\cdot, 0)\|_C^2;$$

moreover, for any $\tilde{u}_0 \in C$ the function $\tilde{u}(x, t)$ tends to 0 exponentially fast, as $t \rightarrow \infty$. On the other hand, if $\tilde{\lambda}_\nu$ is any eigenvalue of \mathcal{H}_* and $\tilde{e}_\nu(x) > 0$ the corresponding normalized eigenfunction then $\tilde{u} = e^{-\tilde{\lambda}_\nu t} \tilde{e}_\nu$ solves (38) with $\tilde{u}_0(x) = \tilde{e}_\nu(x)$. Thus, $\tilde{\lambda}_\nu > 0$ that completes the proof of the claim.

Using Theorem 5, we conclude that for any integers $k \geq 0$ and $l \geq 1$ we can restrict the neighborhoods U_* of $u_*(x, 0)$ and V_0 of 0 in such a way that for any $q \in V_0$ there exists in U_* a unique solution $\tilde{u}(x, q)$ of (36) and the mapping $q \rightarrow \tilde{u}(\cdot, q)$ belongs to class $C^l(V_0, U_*)$. In view of (37), $\tilde{u}(\cdot, q) = u_*(\cdot, q)$ for any $q \in V_0$. \square

3 Appendix: Elliptic equation with parameter

Let $F \times \mathbb{R}^n$ be the product with a compact leaf F , and $g(\cdot, q)$ a leafwise Riemannian metric (i.e., on $F_q = F \times \{q\}$ for $q \in \mathbb{R}^n$) such that the volume form of leaves $d\text{vol}_F = |g|^{1/2} dx$ depends on $x \in F$ only. The Laplacian in a local chart (U, x) on $(F, g|_F)$ is written as $\Delta u = \nabla_i (g^{ij} \nabla_j u) = |g|^{-1/2} \partial_i (|g|^{1/2} g^{ij} \partial_j u)$, see [2] with opposite sign.

This defines an elliptic operator $-\Delta_q$, where $q \in \mathbb{R}^n$ is a parameter and $\Delta_0 = \Delta$,

$$\Delta_q = g^{ij}(x, q) \partial_{ij}^2 + b^j(x, q) \partial_j. \quad (39)$$

Here $b^j = |g|^{-1/2} \partial_i (|g|^{1/2} g^{ij})$ are smooth functions in $U \times \mathbb{R}^n$.

The Schrödinger operator $\mathcal{H}_q = -\Delta_q - \beta(x, q)$ acts in the Hilbert space L_2 with the domain H^2 . Denote $\mathcal{H}_k = \mathcal{H}|_{H^{k+2}}$ and $\mathcal{H}_{q,k} = (\mathcal{H}_q)|_{H^{k+2}}$ for any $q \in \mathbb{R}^n$. Consider also products $\mathbb{B} = L_2 \times \mathbb{R}^n$ and $\mathbb{B}_k = H^k \times \mathbb{R}^n$ (trivial vector bundles over \mathbb{R}^n).

3.1. The Schrödinger operator. If B and C are Banach spaces with norms $\|\cdot\|_B$ and $\|\cdot\|_C$, denote by $\mathfrak{B}^r(B, C)$ the Banach space of all bounded r -linear operators $A : \prod_{i=1}^r B \rightarrow C$ with the norm $\|A\|_{\mathfrak{B}^r(B, C)} = \sup_{v_1, \dots, v_r \in B \setminus 0} \frac{\|A(v_1, \dots, v_r)\|_C}{\|v_1\|_B \cdots \|v_r\|_B}$. If $r = 1$, we shall write $\mathfrak{B}(B, C)$ and $A(\cdot)$, and if $B = C$ we shall write $\mathfrak{B}^r(B)$ and $\mathfrak{B}(B)$, respectively. If M is a k -regular manifold or an open neighborhood of the origin in a real Banach space, and N is a real Banach space, we denote by $C^k(M, N)$ ($k \geq 1$) the Banach space of all C^k -regular functions $f : M \rightarrow N$, for which the following norm is finite:

$$\|f\|_{C^k(M, N)} = \sup_{x \in M} \max\{\|f(x)\|_N, \max_{1 \leq j \leq k} \|d^j f(x)\|_{\mathfrak{B}^j(T_x M, N)}\}.$$

We shall use the simplified version of the Banach's Closed Graph Theorem:

If a linear operator $A : B \rightarrow C$ (of Banach spaces B and C) is bijective and bounded, then its inverse $A^{-1} : C \rightarrow B$ is also bounded.

Lemma 4. *Let $\beta \in C^\infty$ and $\mu < -\max_{x \in F} \beta(x)$. Then*

- (i) $\mathcal{H} - \mu$ acts from H^2 into L_2 , it is continuously invertible and the inverse operator $(\mathcal{H} - \mu)^{-1} : L_2 \rightarrow L_2$ is compact;
- (ii) for any $k \in \mathbb{N}$ the operator $\mathcal{H}_k - \mu$ acts from H^{k+2} into H^k , it is continuously invertible and $(\mathcal{H}_k - \mu)^{-1} : H^k \rightarrow H^k$ is compact;
- (iii) for any integer $k \geq 0$ the spectrum of \mathcal{H}_k , acting in H^k with the domain H^{k+2} , is discrete, and it coincides with the spectrum $\sigma(\mathcal{H})$;
- (iv) for any integer $k \geq 0$ and $\lambda \notin \sigma(\mathcal{H})$ we have

$$R_\lambda(\mathcal{H}_k) = (\mathcal{H}_k - \lambda)^{-1} \in \mathfrak{B}(H^k, H^{k+2}), \quad (40)$$

$$(\lambda \rightarrow R_\lambda(\mathcal{H}_k)) \in C(\mathbb{C} \setminus \sigma(\mathcal{H}), \mathfrak{B}(H^k, H^{k+2})). \quad (41)$$

Proof. (i) Clearly, there exists $C > 0$ such that for any $u \in H^2$ we have

$$\|\mathcal{H}(u) - \mu u\|_{L_2} \leq \|\Delta u - \mu u\|_{L_2} + \|(\beta(x) + \mu)u\|_{L_2} \leq C\|u\|_{H^2}. \quad (42)$$

Thus, $\mathcal{H} - \mu \in \mathfrak{B}(H^2, L_2)$. On the other hand, since $\beta(x) + \mu < 0$, the operator $\mathcal{H} - \mu$ is positive definite as acting in L_2 with the domain H^2 ; hence, it is continuously invertible, i.e., $\mu \notin \sigma(\mathcal{H})$. Thus, $\mathcal{H} - \mu$ maps H^2 injectively onto L_2 . By the Banach's Closed Graph Theorem, $\mathcal{H} - \mu : H^2 \rightarrow L_2$ has the bounded inverse. Compactness of the embedding $H^2 \hookrightarrow L_2$ yields compactness of $(\mathcal{H} - \mu)^{-1} : L_2 \rightarrow H^2$.

(ii) For any $u \in H^{k+2}$ a similar to (42) estimate holds with H^k and H^{k+2} instead of L_2 and H^2 , respectively. Hence, $\mathcal{H}_k - \mu \in \mathfrak{B}(H^{k+2}, H^k)$. By (i), $\mathcal{H} - \mu : H^2 \rightarrow L_2$ is bijective, and

by Elliptic Regularity Theorem, $(\mathcal{H} - \mu)^{-1}(H^k) \subseteq H^{k+2}$ holds. Hence, $\mathcal{H}_k - \mu : H^{k+2} \rightarrow H^k$ is bijective, too. Thus, the Banach's Closed Graph Theorem and the compactness of the embedding $H^{k+2} \hookrightarrow H^k$ complete the proof of (ii).

(iii) Since for any integer $k \geq 0$ the operator $(\mathcal{H}_k - \mu)^{-1} : H^k \rightarrow H^k$ is compact, its spectrum consists of $\nu = 0$ and a countable number of non-zero eigenvalues ν_n of finite multiplicity, which can accumulate only at $\nu = 0$. Hence, the spectrum of \mathcal{H}_k on H_k (with domain in H^{k+2}) is discrete. Since \mathcal{H} extends \mathcal{H}_k , each eigenfunction of \mathcal{H}_k is an eigenfunction of \mathcal{H} ; hence, $\sigma(\mathcal{H}_k) \subseteq \sigma(\mathcal{H})$. To show the opposite inclusion, let $e(x)$ be an eigenfunction of \mathcal{H} , related to $\lambda \in \sigma(\mathcal{H})$ (hence, $e \in H^2$). The obvious equality $e = (\lambda - \nu)^j (\mathcal{H} - \mu)^{-j} e$ is valid for any $j \in \mathbb{N}$. Applying (several times) the Elliptic Regularity Theorem, from $e \in H^2$ we obtain that $e \in H^{k+2}$.

(iv) Similarly to the proof of (ii), we obtain inclusion (40). Let us prove (41). By the well-known property of the resolvent, we have

$$(\lambda \rightarrow R_\lambda(\mathcal{H}_k)) \in C(\mathbb{C} \setminus \sigma(\mathcal{H}_k), \mathfrak{B}(H^k)).$$

Take an arbitrary $\lambda_0 \notin \sigma(\mathcal{H})$ and choose $\delta > 0$ such that the set $D_\delta = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \delta\}$ does not intersect $\sigma(\mathcal{H})$. Using the resolvent identity, see [1],

$$R_\lambda(\mathcal{H}_k) - R_{\lambda_0}(\mathcal{H}_k) = (\lambda - \lambda_0) R_{\lambda_0}(\mathcal{H}_k) R_\lambda(\mathcal{H}_k),$$

we have the following estimate for $\lambda \in D_\delta$:

$$\begin{aligned} \|R_\lambda(\mathcal{H}_k) - R_{\lambda_0}(\mathcal{H}_k)\|_{\mathfrak{B}(H^k, H^{k+2})} &\leq |\lambda - \lambda_0| \times \\ &\times \|R_{\lambda_0}(\mathcal{H}_k)\|_{\mathfrak{B}(H^k, H^{k+2})} \max_{\lambda \in D_\delta} \|R_\lambda(\mathcal{H}_k)\|_{\mathfrak{B}(H^k)}, \end{aligned}$$

which implies the desired inclusion (41). \square

3.2. The ground state. We will show smooth dependence on q of the least eigenvalue $\lambda(q)$ of \mathcal{H}_q and of the corresponding normalized eigenfunction $e(x, q) > 0$.

Lemma 5. *If $\beta \in C^\infty(F \times \mathbb{R}^n)$ then for any $l \in \mathbb{N}$ and integer $k \geq 0$ the mapping $\mathfrak{D} : (u, q) \mapsto \mathcal{H}_q(u)$ is a C^l -morphism (of trivial vector bundles) from \mathbb{B}_{k+2} into \mathbb{B}_k .*

Proof. Fix a finite atlas $\{(U_a, x_a)\}_{1 \leq a \leq A}$ on F , and let $\{\rho_a(x)\}_{1 \leq a \leq A}$ be a subordinated partition of unity. Taking $u \in H^{k+2}$, $q, s \in \mathbb{R}^n$ and using (39), we obtain

$$\begin{aligned} &1/t(\mathfrak{D}(u, q + ts) - \mathfrak{D}(u_a, q)) - \mathfrak{D}_1(u, q) s \\ &= -(1/t \int_0^t \partial_q g^{ij}(x, q + \tau s) s \, d\tau - \partial_q g^{ij}(x, q) s) \partial_{ij}^2 u \\ &\quad - (1/t \int_0^t \partial_q b^i(x, q + \tau s) s \, d\tau - \partial_q b^i(x, q) s) \partial_i u \\ &\quad - (1/t \int_0^t \partial_q \beta(x, q + \tau s) s \, d\tau - \partial_q \beta(x, q) s) u \end{aligned}$$

in a local chart, where

$$\mathfrak{D}_1(\cdot, q) s = -\partial_q g^{ij}(x, q) s \partial_{ij}^2 - \partial_q b^i(x, q) s \partial_i - \partial_q \beta(x, q) s.$$

Hence,

$$\begin{aligned}
& \|1/t(\mathfrak{D}(u, q + ts) - \mathfrak{D}(u, q)) - \mathfrak{D}_1(u, q) s\|_{H^k}^2 = \sum_{a=1}^A \int_{U_a} \rho_a(x) \\
& \sum_{|\vec{m}| \leq k} |\partial_{\vec{m}}^{|\vec{m}|} (1/t(\mathfrak{D}(u, q + ts) - \mathfrak{D}(u, q)) - \mathfrak{D}_1(u, q) s)|^2 dx \\
& \leq C \max_{i,j \in \{1,2,\dots,p\}} \max_{a \in \{1,2,\dots,A\}} \max_{\tau \in [0,t]} (\|\partial_q g^{ij}(x, q + \tau s) - \partial_q g^{ij}(x, q)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))}^2 \\
& + \|\partial_q b^i(x, q + \tau s) - \partial_q b^i(x, q)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))}^2 \\
& + \|\partial_q \beta(x, q + \tau s) - \partial_q \beta(x, q)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))}^2 |s|^2 \|u\|_{H^{k+2}}
\end{aligned}$$

holds for some $C > 0$ that does not depend on u . We conclude that $\mathfrak{D} : H^{k+2} \times \mathbb{R}^n \rightarrow H^k$ has the partial Gateaux differential $\partial_q \mathfrak{D}(u, q)$ at each point (u, q) , and it is equal to $\mathfrak{D}_1(u, q) s$. Similarly, for any $(u_k, q_k) \in H^{k+2} \times \mathbb{R}^n$ ($k = 1, 2$) we obtain

$$\begin{aligned}
& \|\partial_q \mathfrak{D}(u_1, q_1) s - \partial_q \mathfrak{D}(u_2, q_2) s\|_{H^k} \\
& \leq C |s|^2 \max_{i,j \in \{1,2,\dots,p\}} \max_{a \in \{1,2,\dots,A\}} (\|\partial_q g^{ij}(x, q_1) - \partial_q g^{ij}(x, q_2)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))}^2 \\
& + \|\partial_q b^i(x, q_1) - \partial_q b^i(x, q_2)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))} + \|\partial_q \beta(x, q_1) - \partial_q \beta(x, q_2)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))}) \\
& \times \|u_1\|_{H^{k+2}} + (\|\partial_q g^{ij}(x, q_2)\|_{\mathfrak{B}(\mathbb{R}^n)} + \|\partial_q b^i(x, q_2)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))} \\
& + \|\partial_q \beta(x, q_2)\|_{C^k(U_a, \mathfrak{B}(\mathbb{R}^n))}) \cdot \|u_1 - u_2\|_{H^{k+2}},
\end{aligned}$$

and conclude that the partial differential $\partial_q \mathfrak{D}$ is continuous:

$$\partial_q \mathfrak{D} \in C(H^{k+2} \times \mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n, H^k)).$$

One may prove by induction that for any $l \in \mathbb{N}$ the mapping $\mathfrak{D}(\cdot, \cdot)$ has at any point $(u, q) \in H^{k+2} \times \mathbb{R}^n$ the partial differential of l -th order $\partial_q^l \mathfrak{D}(u, q)$, and it has the form

$$\partial_q^l \mathfrak{D}(u, q) = -\partial_q^l g^{ij}(x, q) \partial_{ij}^2 u - \partial_q^l b^i(x, q) \partial_i u - \partial_q^l \beta(x, q) u$$

in a local chart, and $\partial_q^l \mathfrak{D} \in C(H^{k+2} \times \mathbb{R}^n, \mathfrak{B}^l(\mathbb{R}^n, H^k))$. Since $\mathfrak{D}(u, q)$ and $\partial_q^l \mathfrak{D}(u, q)$ are linear by u , this differential is continuous and $\mathfrak{D}(u, q)$ has continuous differentials by q and u of any order. \square

Lemma 6. *Let K be a compact set of $\mathbb{C} \setminus \sigma(\mathcal{H}_{0,k})$ for some integer $k \geq 0$. If $\beta \in C^\infty(F \times \mathbb{R}^n)$, then there is an open neighborhood $W \subseteq \mathbb{R}^n$ of the origin such that*

$$K \subset \mathbb{C} \setminus \sigma(\mathcal{H}_{q,k}) \quad \forall q \in W, \quad (43)$$

and the following inclusion holds for any $l \in \mathbb{N}$:

$$(\lambda \rightarrow R_\lambda(\mathcal{H}_{q,k})) \in C(K, C^l(W, \mathfrak{B}(H^k, H^{k+2}))). \quad (44)$$

Proof. The following obvious representation holds for $\lambda \in K$:

$$\mathcal{H}_{q,k} - \lambda = (\text{id} + L(q, \lambda))(\mathcal{H}_{0,k} - \lambda) \quad (q \in W), \quad (45)$$

where

$$L(q, \lambda) := (\mathcal{H}_{q,k} - \mathcal{H}_{0,k}) R_\lambda(\mathcal{H}_{0,k}). \quad (46)$$

Using Lemma 5, we get that for any integer $l \geq 0$

$$(q \rightarrow \mathcal{H}_{q,k}) \in C^l(W, \mathfrak{B}(H^{k+2}, H^k)). \quad (47)$$

Taking into account Lemma 4(iv), we have for any $(q, \lambda) \in W \times K$

$$\|L(q, \lambda)\|_{\mathfrak{B}(H^k)} \leq \|\mathcal{H}_{q,k} - \mathcal{H}_{0,k}\|_{\mathfrak{B}(H^{k+2}, H^k)} \max_{\mu \in K} \|R_\mu(\mathcal{H}_{0,k})\|_{\mathfrak{B}(H^k, H^{k+2})}.$$

Hence, and in view of (47) with $l = 0$, there exists an open neighborhood $W \subset \mathbb{R}^n$ of the origin such that

$$\sup_{(q, \lambda) \in W \times K} \|L(q, \lambda)\|_{\mathfrak{B}(H^k)} \leq 1/2. \quad (48)$$

Thus, for any $(q, \lambda) \in W \times K$ the operator $\text{id} + L(q, \lambda) \in \mathfrak{B}(H^k)$ is continuously invertible and its inverse is expressed by the Neumann series $(\text{id} + L(q, \lambda))^{-1} = \sum_{j=0}^{\infty} (-L(q, \lambda))^j$ converging in the $\mathfrak{B}(H^k)$ -norm. In view of (45), we conclude that (43) is valid and for any $(q, \lambda) \in W \times K$ we have

$$R_\lambda(\mathcal{H}_{q,k}) = R_\lambda(\mathcal{H}_{0,k})(\text{id} + L(q, \lambda))^{-1}. \quad (49)$$

Lemma 4(iv) and (46)–(47) imply $L(\cdot, \lambda) \in C^l(W, \mathfrak{B}(H^k))$ for $\lambda \in K$ and $l \in \mathbb{Z}$, and using the resolvent identity, we obtain $L(q, \lambda) - L(q, \mu) = (\lambda - \mu)L(q, \mu)R_\lambda(\mathcal{H}_{0,k})$ for $\lambda, \mu \in K$. Hence,

$$\|L(\cdot, \lambda) - L(\cdot, \mu)\|_{C^l(W, \mathfrak{B}(H^k))} \leq |\lambda - \mu| \cdot \|L(\cdot, \mu)\|_{C^l(W, \mathfrak{B}(H^k))} \max_{\nu \in K} \|R_\nu(\mathcal{H}_{0,k})\|_{\mathfrak{B}(H^k)}.$$

This estimate implies

$$(\lambda \rightarrow L(\cdot, \lambda)) \in C(K, C^l(W, \mathfrak{B}(H^k))). \quad (50)$$

By [19, Lemma 7] and the arguments in the end of the proof of [19, Lemma 8], and in view of (48) and (50), we get $(\lambda \rightarrow (\text{id} + L(\cdot, \lambda))^{-1}) \in C(K, C^l(W, \mathfrak{B}(H^k)))$. Then (49) and Lemma 4(iv) imply the desired inclusion (44). \square

Theorem 4. *Let $\lambda(q)$ be the least eigenvalue of \mathcal{H}_q ($q \in \mathbb{R}^n$). If $\beta \in C^\infty(F \times \mathbb{R}^n)$ then $\lambda \in C^\infty(\mathbb{R}^n)$ and there exists a unique smooth section $e : \mathbb{R}^n \mapsto L_2 \times \mathbb{R}^n$ such that $e(\cdot, q)$ is a positive eigenfunction of \mathcal{H}_q related to $\lambda(q)$ with $\|e(\cdot, q)\|_{L_2} = 1$.*

Proof. Assume without loss of generality that $\beta(x, q) < 0$ (otherwise we can consider $\mathcal{H}_q - \mu$ instead of \mathcal{H}_q with a suitable $\mu > 0$). Since $\lambda(q)$ is a simple eigenvalue of \mathcal{H}_q for any $q \in \mathbb{R}^n$, there exists a unique positive eigenfunction $e(\cdot, q)$, related to it, such that $\|e(\cdot, q)\|_{L_2} = 1$. Let λ_0 be the least eigenvalue of the operator \mathcal{H}_0 and e_0 be the eigenfunction related to λ_0 and satisfying conditions mentioned above. Let Γ be a circle of small radius in the complex plane \mathbb{C} not intersecting $\sigma(\mathcal{H}_0)$ and surrounding only λ_0 . By Lemma 6 (with $k = 0$) one may restrict on the open neighborhood Q of 0 in such a way that $\Gamma \subset \mathbb{C} \setminus \sigma(\mathcal{H}_q)$ for any $q \in Q$, and the inclusion (44) is valid with $K = \Gamma$. Hence, in view of $H^2 \hookrightarrow L_2$, the Riesz projection

$$P(q) = -\frac{1}{2\pi i} \oint_{\Gamma} R_\lambda(\mathcal{H}_q) d\lambda \quad (q \in Q) \quad (51)$$

onto the invariant subspace of \mathcal{H}_q corresponding to the part of its spectrum lying inside of Γ ([10, Introduction, Sect. 4]) has the property for any $l \in \mathbb{N}$:

$$P \in C^l(Q, \mathfrak{B}(L_2)). \quad (52)$$

In particular, one may restrict Q in such a way that $\|P(q) - P(0)\|_{\mathfrak{B}(L_2)} \leq 1/2$ for any $q \in Q$. Then, taking into account that $P(q)$ are orthogonal projections (since \mathcal{H}_q are self-adjoint), we have $\dim(\text{Im } P(q)) = \dim(\text{Im } P(0))$, see [1, Chapt. III, Sect. 34]. This means that for $q \in Q$ the operator \mathcal{H}_q has inside of Γ only one simple eigenvalue $\tilde{\lambda}(q)$, it is real because \mathcal{H}_q is self-adjoint, $\text{Im } P(q)$ is the eigenspace of \mathcal{H}_q related to $\tilde{\lambda}(q)$, and $\tilde{\lambda}(0) = \lambda_0$. Denote $\tilde{e}(\cdot, q) = P(q)e_0$. We have for any $q \in Q$

$$\|\tilde{e}(\cdot, q)\|_{L_2} \geq \|P(0)e_0\|_{L_2} - \|(P(q) - P(0))e_0\|_{L_2} \geq 1/2.$$

Thus, $\tilde{e}(x, q)$ is an eigenvector of \mathcal{H}_q related to $\tilde{\lambda}(q)$ such that $\tilde{e}(x, 0) = e_0(x)$. By (52),

$$(q \rightarrow \tilde{e}(\cdot, q)) \in C^l(Q, L_2) \quad (53)$$

for any $l \in \mathbb{N}$. Then the equality $\tilde{\lambda}(q) \langle \mathcal{H}_q^{-1} \tilde{e}(\cdot, q), \tilde{e}(\cdot, q) \rangle_{L_2} = \|\tilde{e}(\cdot, q)\|_{L_2}^2$ and Lemma 6 (with $k = 0$ and $K = \{0\}$) imply that $\tilde{\lambda} \in C^\infty(Q, \mathbb{R})$.

Take an arbitrary $m \in \mathbb{N}$ and set $j = [p/4 + m/2] + 1$. Since the equality $\tilde{e}(\cdot, q) = (\tilde{\lambda}(q))^j \mathcal{H}_q^{-j} \tilde{e}(\cdot, q)$ is valid, then in view of (53) and Lemma 6, we can restrict Q in such a way that $(q \rightarrow \tilde{e}(\cdot, q)) \in C^l(Q, H^{2j})$ for any $l \in \mathbb{N}$. On the other hand, by Sobolev's Embedding Theorem, $H^{2j} \hookrightarrow C^m$, see [2]. Thus, for any $m \in \mathbb{N}$ there is an open neighborhood Q of 0 such that $(q \rightarrow \tilde{e}(\cdot, q)) \in C^\infty(Q, C^m)$. In particular, since $e_0 > 0$, we can restrict Q in such a way that $e(x, q) = \text{Re}(\tilde{e}(x, q)) / \|\text{Re}(\tilde{e}(\cdot, q))\|_{L_2} > 0$ for any $q \in Q$. Clearly, $e(x, q)$ is an eigenfunction of \mathcal{H}_q , related to the eigenvalue $\tilde{\lambda}_q$ and $\|e(\cdot, q)\|_{L_2} = 1$. It remains only to show that it is possible to restrict Q in such a way that $\tilde{\lambda}(q)$ is the least eigenvalue of \mathcal{H}_q for any $q \in Q$, i.e., $\tilde{\lambda}(q) = \lambda(q)$. Indeed, otherwise there is a sequence $q_\nu \in Q$ such that $\lim_{\nu \rightarrow \infty} q_\nu = 0$ and for any ν there exists an eigenvalue $\tilde{\lambda}_\nu$ of \mathcal{H}_{q_ν} obeying $\tilde{\lambda}_\nu < \lambda(q_\nu)$. Since the operators \mathcal{H}_{q_ν} are positive definite and for some $\delta > 0$ in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ there is only the eigenvalue $\tilde{\lambda}(q_\nu)$ of \mathcal{H}_{q_ν} , we get $\tilde{\lambda}_\nu \in [0, \lambda_0 - \delta]$ for any ν . Let $\lambda_* \in [0, \lambda_0 - \delta]$ be a concentration point of the sequence $\{\tilde{\lambda}_\nu\}_{\nu \in \mathbb{N}}$. Choosing a subsequence, we can assume that $\lim_{\nu \rightarrow \infty} \tilde{\lambda}_\nu = \lambda_*$. Surrounding λ_* by a small enough circle Γ such that $\Gamma \cap [\lambda_0, \infty) = \emptyset$, considering for each ν the Riesz projection \tilde{P}_ν , defined by the rhs of (51) with $q = q_\nu$ and using the above arguments, we get that $\lim_{\nu \rightarrow \infty} \|\tilde{P}_\nu - \tilde{P}(0)\|_{\mathfrak{B}(L_2)} = 0$, where $\tilde{P}(0)$ is defined by the rhs of (51) with $q = 0$. Since q_ν lies inside Γ for a large enough ν , $\dim \text{Im} \tilde{P}(0) > 0$. Hence, there is at least one eigenvalue of \mathcal{H}_0 inside of Γ . But this is impossible, because λ_0 is the least eigenvalue of \mathcal{H}_0 . \square

3.3. Solution of the stationary equation. Consider the compact domain in $\mathbb{R} \times F$

$$D := \{(u, x) \in \mathbb{R} \times F : u_-(x) \leq u \leq u_+(x)\}, \quad (54)$$

where $u_-, u_+ \in C^\infty$ and $u_- \leq u_+$. Define sets $G^k = \text{Int}(G) \cap C^k$ for $k > 0$, where $G \subset C$ is a bounded, closed and convex set given by

$$G := \{u \in C : u_-(x) \leq u(x) \leq u_+(x) \text{ for all } x \in F\}. \quad (55)$$

Lemma 7. *Let $m, l \in \mathbb{N}$, Π be an open domain in $\mathbb{R}^m \times F$ of the form $\Pi = \mathbb{R}^{m-1} \times \text{Int}(D)$, and $\theta(\cdot, \cdot, \cdot) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^l$ be a continuous function. Then for any $q \in \mathbb{R}^n$ the superposition operator $\Theta(v, q)(x) = \theta(v(x), x, q)$ maps the set*

$$Y = \{v \in C(F, \mathbb{R}^m) : (v(x), x) \in \Pi \text{ for all } x \in F\}$$

into the set $C(F, \mathbb{R}^l)$, and the inclusion $\Theta \in C(Y \times \mathbb{R}^n, C(F, \mathbb{R}^l))$ holds.

Proof. The first claim is obvious. Let us prove the second one. Suppose that $v_0 \in Y$. Consider the set $\Gamma(v_0) = \{(v_0(x), x)\}_{x \in F}$. Take a relatively compact open set $Q \subset \mathbb{R}^n$ such that $0 \in Q$. In order to construct a similar open neighborhood of the set $\Gamma(v_0)$, observe that it is compact in Π . Then there is a finite open covering $\{U_j\}_{j=1}^k$ of $\Gamma(v_0)$ such that $\overline{U_j} \subset \Pi$ ($1 \leq j \leq k$) and each of $\overline{U_j}$ is compact. Consider the open set $\Pi' = \bigcup_{j=1}^k U_j$. Then $\Gamma(v_0) \subset \Pi'$, $\overline{\Pi'} = \bigcup_{j=1}^k \overline{U_j} \subset \Pi$ and the set $\overline{\Pi'}$ is compact. Consider the following open subset of $C(F, \mathbb{R}^m)$:

$$Y' = \{v \in C(F, \mathbb{R}^m) : (v(x), x) \in \Pi' \text{ for all } x \in F\}.$$

It is clear that $v_0 \in Y'$. Since $\theta(u, x, q)$ is uniformly continuous on the compact $\overline{\Pi'} \times \overline{Q'}$, for any $\epsilon > 0$ there is $\delta > 0$ such that $|\theta(v_1, x, q) - \theta(v_2, x, 0)| < \epsilon$ for all $(v_1, x), (v_2, x) \in \overline{\Omega'}$ and $q \in \overline{Q'}$, where $\|v_1 - v_2\| < \delta$ and $\|q\| < \delta$. Let us choose $\sigma \in (0, \delta)$ such that $B_\sigma(v_0) = \{v \in C(F, \mathbb{R}^m) : \|v - v_0\|_{C(F, \mathbb{R}^m)} < \sigma\} \subset Y'$. If $v \in B_\sigma(u_0)$ and $\|q\| < \delta$ then $\|\Theta(u, q) - \Theta(u_0, 0)\|_{C(F, \mathbb{R}^l)} < \epsilon$ holds. \square

Lemma 8. *Let $f \in C^\infty(D \times \mathbb{R}^n)$. Then the superposition operator*

$$\Phi_f(u, q) = f(u(x), x, q) \quad (56)$$

obeys $\Phi_f \in C^l(G^k \times \mathbb{R}^n, C^k)$ for any integers $k, l \geq 0$.

Proof. This is divided into two steps.

Step 1. First, we shall reduce the operator (56), acting from G^k into C^k to a superposition operator, acting in spaces of continuous vector functions. Take $(u, q) \in G^k \times \mathbb{R}^n$ for some $k \in \mathbb{N}$. Observe that differentials $d^j u(x)$ ($1 \leq j \leq k$) can be considered as functions defined on F with values in \mathbb{R}^{n_j} (e.g., $n_1 = p$, $n_2 = p(p+1)/2$ and so on). We have the following:

$$\begin{aligned} d_x \Phi_f(u, q) &= \partial_x f(u(x), x, q) + \partial_u f(u(x), x, q) du(x), \\ d_x^2 \Phi_f(u, q) &= \partial_x^2 f(u(x), x, q) + \partial_u^2 f(u(x), x, q)(du(x), du(x)) + \partial_u f(u(x), x, q) d^2 u(x), \end{aligned}$$

and so on. Hence,

$$(d_x \Phi_f(u, q), d_x^2 \Phi_f(u, q), \dots, d_x^k \Phi_f(u, q), \Phi_f(u, q)) = \psi(v(x), x, q), \quad (57)$$

where $v(x) = (du(x), d^2 u(x), \dots, d^k u(x), u(x))$ is the vector function on F , and the function $\psi : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^N$ is smooth. Here $\Omega = \mathbb{R}^{N-1} \times \text{Int } D$ and $N = 1 + \sum_{j=1}^k n_j$. It is enough to show that the superposition operator $\Psi(v, q) = \psi(v(x), x, q)$ obeys

$$\Psi \in C^l(X \times \mathbb{R}^n, C(F, \mathbb{R}^N)) \quad (58)$$

for any $l \in \mathbb{N}$, where $X = \{v \in C(F, \mathbb{R}^N) : (v(x), x) \in \Omega \text{ for all } x \in F\}$.

Step 2. In aim to prove (58), take $v \in X$, $q \in \mathbb{R}^n$, $h \in C(F, \mathbb{R}^N)$ and $\delta > 0$ such that $u + th \in X$ for any $t \in [0, \delta]$. We have the following representation for $t \in [0, \delta]$:

$$t^{-1}(\Psi(v + th, q) - \Psi(v, q)) = t^{-1}h(x) \int_0^t \partial_v \psi(v(x) + \tau h(x), x, q) d\tau.$$

Hence,

$$\begin{aligned} &\|t^{-1}(\Psi(v + th, q) - \Psi(v, q)) - \partial_v \psi(v(x), x, q) h(x)\|_{C(F, \mathbb{R}^N)} \leq \\ &\leq \|h\|_{C(F, \mathbb{R}^N)} \sup_{\tau \in [0, t]} \|\partial_v \psi(v + \tau h, \cdot, q) - \partial_v \psi(v, \cdot, q)\|_{C(F, \mathfrak{B}(\mathbb{R}^N))}. \end{aligned}$$

Since for any fixed $q \in \mathbb{R}^n$ the function $g(x, \tau) = \partial_v \psi(v(x) + \tau h(x), x, q)$ is uniformly continuous in $F \times [0, \delta]$, the last estimate implies

$$\lim_{t \downarrow 0} \|t^{-1}g(\Psi(v + th, q) - \Psi(v, q)) - \partial_v \psi(v(x), x, q) h(x)\|_{C(F, \mathbb{R}^N)} = 0.$$

Hence, for any $q \in \mathbb{R}^n$ the operator $\Psi(\cdot, q) : X \rightarrow C(F, \mathbb{R}^N)$ has at any $u \in X$ the Gateaux partial differential $\partial_v \Psi(v, q)$ of the form

$$\partial_v \Psi(v, q) h = \partial_v \psi(v(x), x, q) h(x), \quad h \in C(F, \mathbb{R}^N).$$

We see that $\partial_v \Psi(v, q) \in \mathfrak{B}(C(F, \mathbb{R}^N))$ and

$$\begin{aligned} & \|\partial_v \Psi(v_1, q_1) - \partial_v \Psi(v_2, q_2)\|_{\mathfrak{B}(C(F, \mathbb{R}^N))} \\ & \leq \|\partial_v \psi(v_1, \cdot, q_1) - \partial_v \psi(v_2, \cdot, q_2)\|_{C(F, \mathfrak{B}(\mathbb{R}^N))}. \end{aligned}$$

By Lemma 7 applied to the superposition operator

$$\Psi_1(v, q) = \partial_v \psi(v(x), x, q) : C(F, \mathbb{R}^N) \rightarrow C(F, \mathfrak{B}(\mathbb{R}^N)),$$

the partial differential $\partial_v \Psi(v, q)$ is continuous in the sense that $\partial_v \Psi$ belongs to $C(C(F, \mathbb{R}^N) \times \mathbb{R}^n, \mathfrak{B}(C(F, \mathbb{R}^N)))$. Hence, $\partial_v \Psi(v, q)$ is the Freshet's partial differential. Similarly, one may show that at any $(v, q) \in X \times \mathbb{R}^n$ there is the Gateaux partial differential

$$\partial_q \Psi(v, q) s = \lim_{t \downarrow 0} t^{-1} (\Psi(v, q + ts) - \Psi(v, q)) = \partial_q \psi(v(x), x, q) s,$$

where $s \in \mathbb{R}^n$, and the limit is taken with respect to the $C(F, \mathbb{R}^N)$ -norm. Furthermore, this differential is continuous: $\partial_q \Psi \in C(C(F, \mathbb{R}^N) \times \mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n, C(F, \mathbb{R}^N)))$. Again, this fact implies that $\partial_q \Psi(v, q)$ is the Freshet's partial differential. Thus, we have proved that the superposition operator $\Psi(v, q)(x) = \psi(v(x), x, q)$ belongs to class $C^1(X \times \mathbb{R}^n, C(F, \mathbb{R}^N))$. Applying similar arguments to superposition operators

$$\Psi_1(v, q) = \partial_v \psi(v(x), x, q), \quad \Psi_2(v, q) = \partial_q \psi(v(x), x, q),$$

one may show that $\Psi \in C^2(X \times \mathbb{R}^n, C(F, \mathbb{R}^N))$. Further, we can prove by induction that (58) holds for any $l \in \mathbb{N}$. \square

For $\alpha \in (0, 1)$ and integer $k \geq 0$ consider the Banach space $C^{k, \alpha}$ of such functions $u \in C^k$, for which all the partial derivatives of order k belong to Hölder class $C^{0, \alpha}$. The norm in this space is defined as follows:

$$\|u\|_{C^{k, \alpha}} = \max \left\{ \max_{|\beta| \leq k} \|D^\beta u\|_{C^0}, \max_{|\beta|=k} \sup_{x, y \in F, x \neq y} |D^\beta u(x) - D^\beta u(y)| d(x, y)^{-\alpha} \right\}. \quad (59)$$

Lemma 9. *Let Δ_q be the elliptic operator in (39). Then for any $l \in \mathbb{N}$ the mapping $\mathfrak{D} : (u, q) \mapsto -\Delta_q u$ belongs to class $C^l(C^{k+2, \alpha} \times \mathbb{R}^n, C^{k, \alpha})$.*

Proof. Fix a finite atlas $\{(U_a, x_a)\}_{1 \leq a \leq A}$ of F . Taking $u \in C^{k+2, \alpha}$, $q \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$, we obtain in a local chart, see (39):

$$\begin{aligned} & 1/t(\mathfrak{D}(u, q + ts) - \mathfrak{D}(u, q)) - \mathfrak{D}_1(u, q) s \\ & = -(1/t \int_0^t \partial_q g^{ij}(x, q + \tau s) s \, d\tau - \partial_q g^{ij}(x, q) s) \partial_{ij}^2 u \\ & \quad - (1/t \int_0^t \partial_q b^i(x, q + \tau s) s \, d\tau - \partial_q b^i(x, q) s) \partial_i u, \end{aligned}$$

where $\mathfrak{D}_1(\cdot, q) s = -\partial_q g^{ij}(x, q) s \partial_{ij}^2 - \partial_q b^i(x, q) s \partial_i$. In view of (59),

$$\begin{aligned} & \|1/t(\mathfrak{D}(u, q + ts) - \mathfrak{D}(u, q)) - \mathfrak{D}_1(u, q) s\|_{C^{k, \alpha}(U_a)} \\ & \leq C_a \max_{i, j \in \{1, 2, \dots, p\}} \max_{\tau \in [0, t], x \in U_a} (|\partial_q g^{ij}(x, q + \tau s) - \partial_q g^{ij}(x, q)| \\ & \quad + \|\partial_q b^i(x, q + \tau s) - \partial_q b^i(x, q)\|_{C^{k+1}(U_a, \mathfrak{B}(\mathbb{R}^n))}) |s| \cdot \|u\|_{C^{k+2, \alpha}(U_a)} \end{aligned}$$

holds for some $C_a > 0$ that does not depend on u . Replacing C_a by $C = \max_{1 \leq a \leq A} C_a$, we find that $\mathfrak{D} : C^{k+2, \alpha} \times \mathbb{R}^n \rightarrow C^{k, \alpha}$ has the partial Gateaux differential $\partial_q \mathfrak{D}(u, q)$ at each point (u, q) , and it is equal to $\mathfrak{D}_1(u, q)$. Similarly to the proof of Lemma 5, we obtain that \mathfrak{D} has continuous mixed partial differentials by q and x (of any order) at any point $(u, q) \in C^{k+2, \alpha} \times \mathbb{R}^n$. \square

Theorem 5. Let $f \in C^\infty(D \times \mathbb{R}^n)$ and $u_*(x) \in \text{Int } G$ be a smooth solution of (36) with $q = 0$ such that $\lambda = 0$ is not an eigenvalue of the operator $\mathcal{H} = -\Delta - \partial_u f(u_*(x), x, 0)$ on L_2 with domain in H^2 . Then for any integers $k \geq 0$ and $l \geq 1$ and $\alpha \in (0, 1)$ there are open neighborhoods $U_* \subseteq C^{k+2, \alpha}$ of u_* and $V_0 \subseteq \mathbb{R}^n$ of 0 such for any $q \in V_0$ there exists in U_* a unique solution $u(x, q)$ of (36), in particular, $u_*(x) = u(x, 0)$, such that the function $q \rightarrow u(\cdot, q)$ belongs to class $C^l(V_0, U_*)$.

Proof. By Lemma 8, for any $q \in \mathbb{R}^n$ and integers $k, l \geq 0$ the operator (56) maps the set $G_{k+2} = \text{Int } G \cap C^{k+2}$ into C^{k+2} and $\Phi_f \in C^l(G_{k+2} \times \mathbb{R}^n, C^{k+2})$. Since $C^{k+2, \alpha}$ and C^{k+2} are continuously embedded into C^{k+2} and $C^{k, \alpha}$, respectively, we get $\Phi_f \in C^l(G_{k+2, \alpha} \times \mathbb{R}^n, C^{k, \alpha})$, where $G_{k+2, \alpha} = \text{Int } G \cap C^{k+2, \alpha}$. Consider operators $Y(u, q) = \Delta_q u + \Phi_f(u, q)$ ($q \in \mathbb{R}^n$) defined on $G_{k+2, \alpha}$. By Lemma 9, $Y \in C^l(G_{k+2, \alpha} \times \mathbb{R}^n, C^{k, \alpha})$.

Let $\tilde{\mathcal{H}}$ be \mathcal{H} restricted on $C^{k, \alpha}$ with the domain $C^{k+2, \alpha}$. Set $\beta = \partial_u f(u_*(x), x, 0)$. By (59), there is $C > 0$ such that $\|\Delta u\|_{C^{k, \alpha}} \leq C\|u\|_{C^{k+2, \alpha}}$ for any $u \in C^{k, \alpha}$; hence, $\tilde{\mathcal{H}} - \mu \in \mathfrak{B}(C^{k+2, \alpha}, C^{k, \alpha})$. Let $\mu < -\max_{x \in F} \beta(x)$. By [2, Theorem 4.18], $\mathcal{H}(u) - \mu u = f(x)$ has a unique solution $u \in C^{k+2, \alpha}$ for any $f \in C^{k, \alpha}$, i.e., $\tilde{\mathcal{H}} - \mu$ maps $C^{k+2, \alpha}$ injectively onto $C^{k, \alpha}$. As in the proof of Lemma 4(i), using Banach's Closed Graph Theorem and compactness of the embedding $C^{k+2, \alpha} \hookrightarrow C^{k, \alpha}$, we prove that $\tilde{\mathcal{H}} - \mu$ is continuously invertible and $(\tilde{\mathcal{H}} - \mu)^{-1} : C^{k, \alpha} \rightarrow C^{k+2, \alpha}$ is compact.

By the above, the spectrum of $(\tilde{\mathcal{H}} - \mu)^{-1}$ consists of the point $\nu = 0$ and a countable number of non-zero eigenvalues ν_n of finite multiplicity, which can accumulate only at the point $\nu = 0$. Hence the spectrum of $\tilde{\mathcal{H}}$ is discrete. By Lemma 4(i-iii), $\sigma(\mathcal{H})$ is discrete and $\mu \notin \sigma(\mathcal{H})$. By the same arguments as in the proof of Lemma 4(iii), we find that the spectrum of $\tilde{\mathcal{H}}$ is discrete and coincides with $\sigma(\mathcal{H})$.

Let $\lambda \notin \sigma(\mathcal{H})$. Then $\tilde{\mathcal{H}} - \lambda$ maps injectively $C^{k+2, \alpha}$ into $C^{k, \alpha}$ and it is bounded. By the Elliptic Regularity Theorem, this operator is surjective. By the Banach's Closed Graph Theorem, $\tilde{\mathcal{H}} - \lambda$ acts from $C^{k+2, \alpha}$ into $C^{k, \alpha}$ and it is continuously invertible. Since $0 \notin \sigma(\mathcal{H})$, the partial differential is continuously invertible

$$\partial_u Y(u_*, 0) = -\mathcal{H} \in \mathfrak{B}(C^{k+2, \alpha}, C^{k, \alpha}).$$

These facts and the Implicit Function Theorem, see [2], complete the proof. \square

References

- [1] N.I. Akhiezer and I.M. Glazman, *Theory of linear operators in Hilbert space*, Dover Publications, INC. New York, 1993.
- [2] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer, New-York, 1998.
- [3] G. Bădițoiu, S. Ianus and A.M. Pastore, Spectral geometry of Riemannian Legendre foliations, *Bull. Math. Sci. Math. Romanie*, 10(104), No. 2 (2013), 135–150.
- [4] A. Bejancu and H. Farran, *Foliations and Geometric Structures*, Springer-Verlag, New-York, 2006.
- [5] R.A. Blumenthal, J.J. Hebda, De Rham decomposition theorems for foliated manifolds, *Annales de l'institut Fourier*, 33, No. 2 (1983), 183–198.
- [6] A. Candel and L. Conlon: *Foliations, I, II*, AMS, Providence, 2000.
- [7] B. Chow and D. Knopf, *The Ricci Flow: An Introduction*, AMS, 2004.

- [8] F.Dobarro and B.Ünal, About curvature, conformal metrics and warped products. J. Phys. A: Math. Theor. 40 (2007), 13907-13930.
- [9] P.T.Ho, The long-time existence and convergence of the CR Yamabe flow. Commun. Contemp. Math. 14 (2012), no. 2, 1250014, 50 pp.
- [10] S.G.Krein, *Linear differential equations in Banach space*, Transl. of Math. Monographs, Vol. 29, AMS, Providence, 1972.
- [11] A.M.Naveira, A classification of Riemannian almost-product manifolds, Rend. Mat., 7, no. 3 (1983), 577–592.
- [12] R.Ponge and H.Reckziegel, Twisted products in pseudo-Riemannian geometry, Geom. Dedicata, 48 (1993), 15–25.
- [13] P.Popescu and M.Popescu, On a class of singular vector subbundles, Balkan Society of Geometers Proceedings, 20 (2013), 79–88.
- [14] H.F.Ramírez-Ospina, Multiplicity of constant scalar curvature metrics in $T^k \times M$. Non-linear Anal. 109 (2014), 103–112.
- [15] V.Rovenski, On solutions to equations with partial Ricci curvature, J. of Geometry and Physics, 86 (2014), 370–382.
- [16] V.Rovenski, *Foliations on Riemannian Manifolds and Submanifolds*, Birkhäuser, Boston, 1998.
- [17] V.Rovenski and P.Walczak, *Topics in Extrinsic Geometry of Codimension-One Foliations*, Springer Briefs in Math., Springer, New-York, 2011.
- [18] V.Rovenski and R.Wolak, Deforming metrics of foliations, Cent. Eur. J. Math. 11 (2013), 1039–1055.
- [19] V.Rovenski and L.Zelenko, Prescribing the positive mixed scalar curvature of totally geodesic foliations, 163–203, in “*Foliations-2012*”, World Sci., Singapore 2013.
- [20] V.Rovenski and L.Zelenko, Prescribing the mixed scalar curvature of a foliation, pp. 79–110, in “Geometry and its Applications”, Springer Proc. in Mathematics & Statistics, 72, Springer, New-York, 2014.
- [21] P.Schweitzer and P.Walczak, Prescribing mean curvature vectors for foliations, Illinois J. Math. 48 (2004), 21–35.
- [22] P.Walczak, An integral formula for a Riemannian manifold with two orthogonal complementary distributions, Colloq. Math., 58 (1990), 243–252.
- [23] G.Wang and Y.Zhang, A conformal integral invariant on Riemannian foliations, Proc. Amer. Math. Soc., 141, No. 4 (2013), 1405–1414.
- [24] R.Takagi and S.Yorozu, Minimal foliations on Lie groups, Tohoku Math. J. 36 (1984), 541–554.
- [25] W.Zhang, A Lichnerowicz-Hitchin vanishing theorem for foliations, 53 pp., arXiv:1204.2224.